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Inégalités d'Arakelov et familles semistable de courbes uniformisées par la boule

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Titre Inégalités d'Arakelov et familles semistables de courbes uniformisées par la boule

Résumé L'objet principal de cette thèse est de démontrer une inégalité d'Arakelov qui consiste à borner le degré d'un sous-faisceau inversible de l'image directe d'un faisceau relatif pluricanonique d'une famille semi-stable de courbes. Un problème naturel qui apparaît est la caractérisation des familles pour lesquelles sont satisfaites le cas d'égalité dans l'inégalité d'Arakelov, i.e. le cas d'égalité d'Arakelov. Peu d'exemples de telles familles sont connus. Dans cette thèse nous en proposons plusieurs en prouvant que le faisceau relatif bi-canonique d'une famille semi-stable de courbes uniformisée par la boule unité et dont toutes les fibres singulières sont totalement géodésiques contient un sous-faisceau inversible qui satisfait l'égalité d'Arakelov.

Title Arakelov inequalities and semistable families of curves uniformized by the unit ball

Abstract The main object of study in this thesis is an Arakelov inequality which bounds the degree of an invertible subsheaf of the direct image of the pluricanonical relative sheaf of a semistable family of curves. A natural problem that arises is the characterization of those families for which the equality is satisfied in that Arakelov inequality, i.e. the case of Arakelov equality. Few examples of such families are known. In this thesis we provide some examples by proving that the direct image of the bicanonical relative sheaf of a semistable family of curves uniformized by the unit ball, all whose singular fibers are totally geodesic, contains an invertible subsheaf which satisfies Arakelov equality.

Keywords Cyclic coverings, Semistable families of curves, Variations of Hodge structures, Higgs bundles, Ball quotients, Teichmüller curves

Mots-clés Revêtements cycliques, Familles de courbes semi-stables, Variations de structures de Hodge, Fibrés de Higgs, Quotients de la boule, Courbes de Teichmüller

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Introduction

Un bref aperçu historique

Arakelov a montré dans son papier [1] qu'il n'y a, à isomorphisme près, qu'un nombre fini de familles $f : X \to Y$ de courbes algébriques de genre fixé plus grand que 1 au-dessus d'une courbe projective lisse Y de lieu singulier S, qui est un ensemble fini de points sur la courbe de base Y. La preuve du théorème d'Arakelov consiste en deux parties.

Dans un premier temps il a montré que l'ensemble des familles lisses

$$\{f : W \to Y \backslash S\}$$

et l'ensemble des sections

$$\{s : Y \backslash S \to W | f \circ s = id_{Y \backslash S}\}$$

des familles f sont en nombre fini.

Dans un second temps, il a prouvé la rigidité, c'est-à-dire qu'il a montré que pour une famille donnée $f : W \to Y \backslash S$ et une section $s : Y \backslash S \to W$, il n'est pas possible de déformer f ou s.

Un problème similaire a été traité par Faltings dans [25] pour des familles de variétés abéliennes polarisées pour lesquelles il a prouvé que la finitude est satisfaite en général mais que la rigidité est satisfaite si tous les endormorphismes du système local des premiers groupes de comologie de fibres proviennent des endomorphismes des variétés abéliennes.

Ses résultats consistent à montrer que le degré de l'image directe du faisceau canonique relatif d'une famille est borné. Ainsi l'inégalité d'Arakelov initiale peut être formulée comme suit :

Soit $f : X \to Y$ une famille semistable de courbes de fibre générique de genre g, dont le lieu singulier est S, alors on obtient:

$$\deg f_* \omega_{X/Y} \leq \frac{g}{2} \deg \Omega^1_Y(\log S). \tag{1}$$

Tan a montré dans son papier [71] que l'inégalité précédente est stricte pour les familles de courbes de fibres de genre supérieur à 1. Une des principales conséquences de ce résultat est la réponse à la question de Szpiro posée dans [70]:

Soit $f : X \to \mathbb{P}^1$ une famille semi-stable de courbes de genre g, qui n'est pas isotriviale. Quel est le nombre minimal de fibres singulières de cette famille? Ce problème a été traité pour la première fois par Beauville dans [5], où il a prouvé le théorème suivant:

Théorème A *Soit $f : X \to \mathbb{P}^1$ une famille semi-stable de courbes de genre $g \geq 1$, qui n'est pas isotroviale. Alors cette famille f possède au moins 4 fibres singulières.*

Après avoir donné plusieurs exemples de fibrations elliptiques avec 4 fibres singulières dans [6], ainsi que plusieurs exemples de familles dont les fibres ont un genre supérieur à 1, Beauville a stipulé la conjecture suivante:

Conjecture de Beauville *Soit $f : X \to \mathbb{P}^1$ une famille semi-stable de courbes de genre $g > 1$, qui n'est pas isotroviale. Alors cette famille possède au moins 5 fibres singulières.*

En combinant les résultats de Beauville et le fait que l'inégalité d'Arakelov est stricte pour $g > 1$, Tan a pu démontrer cette conjecture.

Une approche plus générale des inégalités d'Arakelov a été reprise par Peters, Jost et Zuo. En effet, ils ont considéré une variation réelle polarisée de structures de Hodge de poids k sur une courbe lisse $Y \backslash S$ où S est un ensemble fini de points sur une courbe projective lisse Y. Il est connu qu'un système de fibrés de Hodge $E = \bigoplus_{p+q=k} E^{p,q}$ d'application $\Theta^{p,q} : E^{p,q} \to E^{p+1,q-1} \otimes \Omega^1_{Y\backslash S}$ peut être associé à cette variation, dont l'extension $\overline{E} = \bigoplus_{p+q=k} \overline{E}^{p,q}$ d'application $\overline{\Theta}^{p,q} : \overline{E}^{p,q} \to \overline{E}^{p-1,q+1} \otimes \Omega^1_Y(\log S)$ sur Y est quasi-canonique si les monodromies du système local de la variation sont quasi-unipotentes. Elle est canonique si les monodromies sont unipotentes. Dans [58] et [39], les auteurs ont donné des inégalités qui bornent le degré du plus haut fibré $\bar{E}^{k,0}$ de l'extension quasi-canonique (ou canonique) du système de fibrés de Hodge.

On peut immédiatement voir que pour la variation géométrique associée à la famille semi-stable $f : X \to Y$ de courbes de genre g, lisse au-dessus $Y \backslash S$, on obtient l'extension canonique $\overline{E} = f_*\omega_{X/Y} \oplus R^1 f_* \mathcal{O}_X$ du système de fibré de Hodge associée à la variation géométrique, d'application $\overline{\Theta} : f_*\omega_{X/Y} \to R^1 f_* \mathcal{O}_X \otimes \Omega^1_Y(\log S)$ et ainsi

$$\deg f_*\omega_{X/Y} \leq \frac{g - h_0^{1,0}}{2} \deg \Omega^1_Y(\log S). \tag{2}$$

Cela vient du fait que $h^{1,0} = \mathrm{rank}\ f_*\omega_{X/Y} = g.$. En comparant les inégalités (1) et (2), il apparaît que l'inégalité (2) est plus forte que l'inégalité d'Arakelov initiale. Le cas intéressant à considérer est celui où l'inégalité précédente devient une égalité. En particulier, c'est la cas quand l'application $\bar{\Theta}$ est un isomorphisme. Ainsi, par le résultat de Tan, ceci est seulement possible pour les familles de courbes de genre 1.

Une autre classe d'inégalité d'Arakelov fabriquée par Viehweg et Zuo est celle qui donne des bornes sur le degré des faisceaux relatifs pluricanonique de familles semi-stables de variétés lisses au-dessus d'une courbe projective lisse. En effet, ils ont obtenu des inégalités sur le degré de $f_*\omega_{X/Y}^{\nu}$, où ν est un entier positif et $f : X \to Y$ est une famille semi-stable de n-variétés lisses au-dessus d'une courbe projective lisse Y de lieu singuliere un ensemble fini S sur Y. Leur principal résultat, qui peut être trouvé dans [55], est le suivant:

$$\frac{\deg f_*\omega_{X/Y}^{\nu}}{\text{rank } f_*\omega_{X/Y}^{\nu}} \leq \frac{n\nu}{2} \deg \Omega_Y^1(\log S). \tag{3}$$

Ce résultat est la conséquence d'une autre inégalité d'Arakelov qui borne le degré d'un sous-faisceau inversible de $f_*\omega_{X/Y}^{\nu}$.. Pour n'importe quel sous-faisceau inversible $\mathcal{H}$ de $f_*\omega_{X/Y}^{\nu}$., Viehweg et Zuo ont démontré dans [55] que

$$\deg \mathcal{H} \leq \frac{n\nu}{2} \deg \Omega_Y^1(\log S). \tag{4}$$

Le principal objet de cette thèse concerne cette dernière inégalité. Si on suppose que $f : X \to Y$ est une famille semi-stable de courbes et que le cas d'égalité dans (4) est vérifiée, alors pour $\nu = 1$, en utilisant un résultat de Möller de [52], on obtient que $Y \backslash S$ est une courbe de Teichmüller. Une question naturelle est la caractérisation des familles de courbes pour lesquelles l'égalité dans (4) est vérifiée quand $\nu \geq 2$.

Organisation de la thèse

Cette thèse est divisée en quatre chapitres. Les deux premiers chapitres sont de nature introductive, contenant des résultats connus et leurs preuves légèrement modifiées. Le troisième chapitre apporte un nouveau point de vue sur la preuve de l'inégalité (4) et quelques remarques pour le cas d'égalité dans (4). Le dernier chapitre contient de nouveaux résultats en proposant des exemples de familles pour lesquelles l'égalité dans (4) est vérifiée.

Le premier chapitre présente les définitions de base et des résultas sur les revêtements cycliques de n-variétés, les formes différentielles logarithmiques et la cohomologie des revêtements cycliques, donnés par Hélène Esnault et Eckart Viehweg dans [20]. Certains résultats sont expliqués dans leur forme initiale et d'autres sont prouvés à nouveaux et adaptés au reste de la thèse.

Dans le second chapitre, nous rappelons les définitions et les constructions des fibrés de Higgs et des fibrés logarithmiques de Higgs associés aux variations de structures de Hodge sur des courbes compactes et non-compactes, ainsi que plusieurs résultats de Deligne et Simpson. Dans la dernière section de ce chapitre, nous donnons un aperçu des résultats de base sur les espaces de Teichmüller et les courbes de Teichmüller, et nous développons un résultat sur la connexion entre les variations polarisées de rang 2 et de poids 1 et les courbes de Teichmüller.

Le troisième chapitre est le plus important d'un point de vue technique. Nous y expliquons la preuve de Viehweg et Zuo sur l'inégalité d'Arakelov (4). Leur résultat fondamental est appliqué pour une famille semi-stable de n-variétés, en revanche dans cette thèse nous modifions et procurons tous les détails de preuve pour des familles de courbes semi-stable. Dans la dernière section de ce chapitre, nous discutons du cas d'égalité dans (4), que l'on appelle le cas maximal. Nous esperions prouver que même dans le cas $\nu \geq 2$, $Y \backslash S$ est une courbe de Teichmüller. Malheureusement, nous n'avons pas réussi à le vérifier et nous pouvons seulement donner quelques informations partielles sur cette famille.

Le dernier chapitre de la thèse contient de nouveaux résultats. Notamment, des exemples naturelles de familles semi-stables de courbes dont l'image directe du faisceau relatif pluricanonique contient un sous-faisceau inversible réalisant l'égalité dans (4) ne sont pas connues, mis à part les cas évidents qui sont construits en partant d'une situation où $\nu = 1$. Il se trouve que pour une famille de courbes uniformisées par la boule, il existe un sous-faisceau inversible naturel de l'image directe du faisceau relatif bicanonique construit à partir de la seconde forme fondamentale de la famille. Nous expliquons ce concept en utilisant le résultat de Mok de [53] concernant les secondes formes projectives fondamentales et les feuilletages tautologiques de la projectivisation du fibré tangent de l'espace des formes hyperboliques complexes, qui sont les quotients de la n-boule unité complexe $\mathbb{B}^n$ par des sous-groupes discrets, co-compacts et sans torsion de $\mathrm{PU}(n, 1)$.

Théorème B *Soit $f : X = \mathbb{B}^2/\Gamma \to Y$ une famille semi-stable de courbes, où Γ est un sous-groupe discret, co-compact et sans torsion de $\mathrm{PU}(2, 1)$, telle que la famille est lisse au-dessus de $Y \backslash S$ avec toutes ses fibres singulières totalement géodésiques, et le genre de Y est supérieur à 1. Alors, il existe un sous-faisceau inversible de l'image directe du faisceau relatif bicanonique $f_*\omega_{X/Y}^{\otimes 2}$ qui satisfait l'égalité d'Arakelov* (4).

Guidés par cette idée et en utilisant les résultats de Livné dans [49], nous donnons plusieurs exemples de familles semi-stables de courbes uniformisées par la 2-boule complexe au-dessus des courbes modulaires de niveaux $N \in \{7, 8, 9, 12\}$. Nous démontrons que toutes les fibres singulières dans ces familles sont totalement géodésiques, et nous prouvons ensuite que ce sont des exemples de familles dont la courbe de base privée du lieu singulier de la famille est une courbe de Teichmüller. Cependant, il est intéressant de remarquer que ces exemples pour lesquels l'image directe du faisceau relatif bicanonique contient un sous-faisceau maximal sont aussi construits à partir de cas maximaux pour $\nu = 1$.

Une question qui se pose toujours est de savoir si n'importe quelle famille décrite dans le Théorème B fournit un exemple de courbe de Teichmüller.

A short historical overview

Arakelov showed in his paper [1] that there are only finitely many families $f : X \to Y$ of algebraic curves up to isomorphism of fixed genus bigger than 1 over a projective smooth curve Y with fixed discriminant locus S, which is a finite set of points on the base curve. The proof of Arakelov's theorem consists in two parts:

In the first part he showed the boundedness, i.e. he showed that the set of all smooth families

$$\{f : W \to Y \backslash S\}$$

and the set of all sections

$$\{s : Y \backslash S \to W | f \circ s = id_{Y \backslash S}\}$$

of families f, consist in a finite number of families.

In the second part, he proved the rigidity, where he showed that if one has a given family $f : W \to Y \backslash S$ and a section $s : Y \backslash S \to W$, then it is not possible to deform f or s.

A similar problem was treated by Faltings in [25] for families of polarized abelian varieties, where he proved that the boundedness holds in general, but the rigidity holds if all endomorphisms of the local system of the first cohomology groups of fibers come from endomorphisms of the abelian varieties.

These results treat the boundedness of the degree of the direct image of the relative canonical sheaf of a family. So, the original Arakelov inequality can be formulated as:

Given a semistable family of curves $f : X \to Y$ whose generic fiber has genus g, with discriminant set S, one has:

$$\deg f_* \omega_{X/Y} \leq \frac{g}{2} \deg \Omega^1_Y(\log S). \tag{5}$$

Tan proved in his paper [71] that the previous inequality is strict for the families of curves whose fibers have genus bigger or equal to 2. One of the main consequences of this result was the answer to Szpiro's question in [70]:

Let $f : X \to \mathbb{P}^1$ be a semistable family of curves of genus g, which is not isotrivial. What is the minimal number of singular fibers of the family? This problem was firstly treated by Beauville in [5], where he proved the theorem:

Theorem 0.1. *Let $f : X \to \mathbb{P}^1$ be a semistable family of curves of genus $g \geq 1$, which is not isotrivial. Then, the family has at least 4 singular fibers.*

After he gave several examples of elliptic fibrations with 4 singular fibers in [6] and several examples of families whose fibers have genus bigger than 1 and the discriminant locus of these families has cardinality bigger than 4 he stated the conjecture:

Conjecture 0.2. *(Beauville's conjecture) Let $f : X \to \mathbb{P}^1$ be a semistable family of curves of genus $g > 1$ which is not isotrivial. Then, the family has at least 5 singular fibers.*

Combining Beauville's results and the fact the Arakelov inequality is strict for $g > 1$, Tan proved the conjecture.

A more general approach to the class of Arakelov inequalities was taken by Peters, Jost and Zuo. In fact, they considered a real polarized variation of Hodge structures of weight k on a smooth curve $Y \backslash S$ where S is a finite set of points on a smooth projective curve Y. It is well known that one has a system of Hodge bundles $E = \bigoplus_{p+q=k} E^{p,q}$ with maps $\Theta^{p,q} : E^{p,q} \to E^{p+1,q-1} \otimes \Omega^1_{Y \backslash S}$ associated to this variation, whose extension $\overline{E} = \bigoplus_{p+q=k} \overline{E}^{p,q}$ with maps $\overline{\Theta}^{p,q} : \overline{E}^{p,q} \to \overline{E}^{p-1,q+1} \otimes \Omega^1_Y(\log S)$ to Y is quasi-canonical if monodromies of the local system of the variation are quasi-unipotent and it is canonical if the monodromies are unipotent. In [58] and [39] they gave inequalities which bound the degree of the top bundle $\overline{E}^{k,0}$ of the quasi-canonical (canonical) extension of the system of Hodge bundles:

One can see immediately that for the geometric variation associated to a semistable family $f : X \to Y$ of curves of genus g, smooth over $Y \backslash S$ one gets the canonical extension $\overline{E} = f_*\omega_{X/Y} \oplus R^1 f_* \mathcal{O}_X$ of the system of Hodge bundles associated to the geometric variation, with the map $\overline{\Theta} : f_*\omega_{X/Y} \to R^1 f_* \mathcal{O}_X \otimes \Omega^1_Y(\log S)$ and therefore:

$$\deg f_*\omega_{X/Y} \leq \frac{g - h_0^{1,0}}{2} \deg \Omega^1_Y(\log S). \tag{6}$$

It holds since $h^{1,0} = \text{rank } f_*\omega_{X/Y} = g$. Comparing the inequality (6) with the inequality (5), one gets that the inequality (6) is stronger than the original Arakelov inequality. The interesting case to be considered is the one when the previous inequality becomes an equality. And that is the case when the map $\overline{\Theta}$ is an isomorphism. Therefore, by Tan's results that is only possible for the families of curves of genus equal to 1.

Another class of Arakelov inequalities made by Viehweg and Zuo is the one which gives bounds on the degree of pluricanonical relative sheaves of semistable families of manifolds over a projective smooth curve. In fact, they produced inequalities about the degree of $f_*\omega^{\nu}_{X/Y}$, where ν is a positive integer and $f : X \to Y$ is a semistable family of n-manifolds over a smooth projective curve Y and with discriminant locus a finite set S on Y. The main result, that can be found in [55], is the following:

$$\frac{\deg f_*\omega^{\nu}_{X/Y}}{\text{rank } f_*\omega^{\nu}_{X/Y}} \leq \frac{n\nu}{2}\deg \Omega^1_Y(\log S). \tag{7}$$

This result is the consequence of one other Arakelov inequality which bounds the degree of an invertible subsheaf of $f_*\omega^{\nu}_{X/Y}$. For any invertible subsheaf $\mathcal{H}$ of $f_*\omega^{\nu}_{X/Y}$, Viehweg and Zuo proved in [55] that:

$$\deg \mathcal{H} \leq \frac{n\nu}{2}\deg \Omega^1_Y(\log S). \tag{8}$$

The main object of the study in this thesis is the last inequality. If one supposes that $f : X \to Y$ is a semistable family of curves and that equality holds in (8), then for $\nu = 1$ by using a result by Möller from [52], one gets that $Y\backslash S$ is a Teichmüller curve. A natural problem that arises is the characterization of the families of curves for which the equality in (8) holds when $\nu \geq 2$.

Organization of the thesis

This thesis is divided in four chapters. The first two chapters are of an introductory nature, containing some well known results with slightly modified proofs. The third chapter brings some new point of view on the proof of the inequality (8) and some remarks for the case of equality in (8). The last chapter contains original results, providing some examples of families where the equality in (8) is obtained.

The first chapter contains some basic definitions and results about cyclic coverings on n-manifolds, logarithmic differential forms and cohomology of cyclic cyclic coverings, as given by Hélène Esnault and Eckart Viehweg in [20]. Some results are explained in their original form and others are reproved and adapted to the rest of the thesis.

In the second chapter we recall the definitions and constructions of Higgs bundles and logarithmic Higgs bundles associated to variations of Hodge structures on compact and non-compact curves, and several important results of Deligne and Simpson. In the last section of this chapter, we give a short review of basic facts about the Teichmüller space and Teichmüller curves, and we explain one result of Möller about the connection between polarized variations of rank 2 and weight 1 with Teichmüller curves.

The third chapter is the most important chapter in the technical sense. There we explain the proof by Viehweg and Zuo of the Arakelov inequality (8). Their original result is done for a semistable family of n-manifolds, but in this thesis we modify and fill in all details of the proof for families of semistable curves. In the last section of that chapter, we discuss the case of equality in

(8), which is called the maximal case. Our expectations was to prove that even when $\nu \geq 2$, $Y \backslash S$ is a Teichmüller curve. Unfortunately, we were not able to prove that and we can only give some partial information on the family.

The last chapter of the thesis contains original results. It seems that examples of semistable families of curves whose direct image of the pluricanonical relative sheaf contains an invertible subsheaf realizing the equality in (8) are not known, except in the obvious cases which are constructed starting from a situation with $\nu = 1$. It happens that for a family of curves uniformized by the ball, there exists a natural invertible subsheaf of the direct image of the bicanonical relative sheaf constructed from the second fundamental form of the family. We explain this fact using Mok's result from [53] about second projective fundamental forms and tautological foliations of the projectivization of the tangent bundle of complex hyperbolic space forms, which are quotients of the unit complex n-ball $\mathbb{B}^n$ by discrete, co-compact, torsion free subgroups of $\mathrm{PU}(n, 1)$.

Theorem 0.3. *Let $f : X = \mathbb{B}^2/\Gamma \to Y$ be a semistable family of curves, where Γ is a discrete, co-compact, torsion free subgroup of $\mathrm{PU}(2, 1)$, the family is smooth over $Y \backslash S$ with all singular fibers totally geodesic and the genus of Y is bigger than 1. Then, there exists an invertible subsheaf of the direct image of the bicanonical relative sheaf $f_*\omega_{X/Y}^{\otimes 2}$ which satisfies the Arakelov equality (8).*

After that, guided by this idea and using the results of Livné from [49], we provide several examples of semistable families of curves uniformized by the complex 2-ball over modular curves of level $N \in \{7, 8, 9, 12\}$. We prove that all singular fibers in these families are totally geodesic, and then we prove that these are examples of families whose curve of the base without the discriminant locus of the family is a Teichmüller curve. However, notice that these examples where the direct image of the relative bicanonical sheaf contains a maximal subsheaf are again constructed from maximal cases for $\nu = 1$.

A question that still remains to be answered is whether any family as in Theorem 0.3 provides an example of Teichmüller curve.

Chapter 1

Cyclic coverings

1.1 Constructions of cyclic coverings

In this section we describe the construction of cyclic coverings over manifolds. We will explain the algebraic construction and relate it to the geometric point of view. This construction relies on Exercise 5.17 from [32] §II. In the first part of the section we will recall some definitions and certain categorical equivalences. In the second part we will recall basic properties of cyclic coverings. The construction and the results are taken from [20] §3.

Definition 1.1. Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules, or simply an $\mathcal{O}_X$-module, is a sheaf $\mathcal{F}$ of abelian groups on X, such that for each open set $U \subseteq X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for each inclusion of open sets $V \subseteq U$, the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structure via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. Any direct sum or direct product of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module. An $\mathcal{O}_X$-module $\mathcal{F}$ is free if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$.

A sheaf of $\mathcal{O}_X$-algebras is a sheaf $\mathcal{A}$ of abelian groups on X such that for each open set $U \subseteq X$, the group $\mathcal{A}(U)$ is an $\mathcal{O}_X(U)$-algebra and for each inclusion of open sets $V \subseteq U$, the restriction homomorphism $\mathcal{A}(U) \to \mathcal{A}(V)$ is a morphism of rings, and for $r \in \mathcal{O}_X(U)$ and $m \in \mathcal{A}(U)$ we have $(r \cdot m)_{|V} = r_{|V} \cdot m_{|V}$.

Definition 1.2. Let R be a commutative ring and M be an R-module. Let $T^n(M)$ be the tensor product $M \otimes M \otimes ... \otimes M$ of M with itself n times, for $n \geq 1$. For $n = 0$ we put $T^0 = R$. Then $T(M) = \bigoplus_{n \geq 0} T^n(M)$ is an $R-$algebra, that we call the tensor algebra of M.

Definition 1.3. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The tensor algebra of $\mathcal{F}$ is defined by taking the sheaf associated to the presheaf, which to each open set U assigns the tensor algebra of $\mathcal{F}(U)$ as an $\mathcal{O}_X(U)$-module. The results are graded $\mathcal{O}_X$-algebras, their components in each degree are $\mathcal{O}_X$-modules.

Let us now recall the definitions of finite type, resp. quasicoherent, resp. finitely presented and coherent $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, let U be an open set of X and $s \in \mathcal{F}(U)$ a section, then there is the unique map associated to s defined as:

$$\mathcal{O}_U \to \mathcal{F}_{|U}, \quad f \to f \cdot s. \tag{1.1}$$

A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is of *finite type* if for every point $x \in X$ there exists an open neighborhood $x \in U \subset X$ such that $\mathcal{F}_{|U}$ is generated by finitely many sections. It is *quasicoherent* if for every point $x \in X$ there exists an open neighborhood $x \in U \subset X$ such that $\mathcal{F}_{|U}$ is isomorphic to the cokernel of a map:

$$\bigoplus_{j \in J} \mathcal{O}_U \to \bigoplus_{i \in I} \mathcal{O}_U.$$

In other words, there is an open covering on X by sets U such that $\mathcal{F}_{|U}$ can be presented as:

$$\bigoplus_{j \in J} \mathcal{O}_U \to \bigoplus_{i \in I} \mathcal{O}_U \to \mathcal{F}_{|U} \to 0.$$

If moreover the sets I and J are finite then $\mathcal{F}_{|U}$ is *finitely presented.* It is a *coherent* sheaf if it is of finite type and for every open set U and every finite collection of sections $s_i \in \mathcal{F}(U), i = 1, ..., n$ the kernel of the associated map induced by (1.1) $\bigoplus_{i=1,\dots n} \mathcal{O}_U \to \mathcal{F}_{|U}$ is of finite type. A coherent sheaf is finitely presented and in particular it is quasicoherent. Also, if X is a Noetherian scheme then the following are equivalent:

- $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module;
- $\mathcal{F}$ is a quasicoherent $\mathcal{O}_X$-module of finite type;
- $\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module.

Remark 1.4. Let us recall that an $\mathcal{O}_X$-module $\mathcal{F}$ on a ringed space $(X, \mathcal{O}_X)$ is invertible if for every $x \in X$ there is a neighborhood U of x such that $\mathcal{F}_{|U}$ is isomorphic to $\mathcal{O}_U$ as an $\mathcal{O}_U$-module. An invertible $\mathcal{O}_X$-module on $(X, \mathcal{O}_X)$ is quasicoherent.

Definition 1.5. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be an $\mathcal{O}_X$-algebra. It is of finite type, resp. quasicoherent, finitely presented or coherent if it has these properties as a sheaf of $\mathcal{O}_X$-modules.

Theorem 1.1. *([32],p.128) Let $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra on a scheme X. Then there is a unique scheme W and a unique morphism $\tau : W \to X$ such that for every open affine $U \subseteq X$, $\tau^{-1}(U) = Spec(\mathcal{A}(U))$ and for every inclusion $V \hookrightarrow U$ of open affines in X, the morphism $\tau^{-1}(V) \hookrightarrow \tau^{-1}(U)$ corresponds to the restriction homomorphism $\mathcal{A}(U) \to \mathcal{A}(V)$. We denote the scheme W as $Spec_X(\mathcal{A})$.*

Since we will work in the category of projective complex manifolds, we should recall that this category is equivalent to the category of regular projective reduced schemes over $\mathbb{C}$. This is a consequence of Chow's theorem. Chow's theorem states that a closed submanifold of projective space is closed in the Zariski topology and it defines canonically a regular projective reduced scheme over $\mathbb{C}$. On the other side Serre's GAGA theorem provides the equivalence of categories of algebraic coherent sheaves on X, when X is a regular projective reduced scheme over $\mathbb{C}$, with the category of coherent analytic sheaves on X, when X is seen as a projective complex manifold. We should also add the fact that projective schemes over $\mathbb{C}$ are Noetherian.

This enables us to apply Theorem 1.1 in the setting of projective complex manifolds.

Let X be a projective complex manifold and let $D = \sum_{j=1}^{k} \alpha_j D_j$ be an effective divisor on X. Let ν be a positive integer and $\mathcal{L}$ be an invertible sheaf such that:

$$\mathcal{L}^{\nu} \simeq \mathcal{O}_X(D).$$

Once and for all let us fix such an isomorphism. Let $s \in H^0(X, \mathcal{L}^{\nu})$ be a section whose zero divisor is D. The dual of

$$s : \mathcal{O}_X \to \mathcal{L}^{\nu},$$

i.e.

$$s^{\vee} : \mathcal{L}^{-\nu} \to \mathcal{O}_X$$

defines an $\mathcal{O}_X$-algebra:

$$\mathcal{A}' = \bigoplus_{i=0}^{\nu-1} \mathcal{L}^{-i},$$

where the multiplication in $\mathcal{A}'$ is defined by

$$\mathcal{L}^{-i} \times \mathcal{L}^{-j} \to \mathcal{L}^{-i-j},$$

composed with $s : \mathcal{L}^{-i-j} \to \mathcal{L}^{-i-j+\nu}$ in case $i + j \geq \nu$.

Definition 1.6. Let E be a $\mathbb{Q}$-divisor, then the divisor $[E]$ denotes the divisor whose multiplicities of components are the integral parts of the multiplicities of the components of the divisor E.

Now, let us define for any non-negative integer i,

$$\mathcal{L}^{(-i)} := \mathcal{L}^{-i} \otimes \mathcal{O}_X\left(\left[\frac{iD}{\nu}\right]\right)$$

and

$$\mathcal{A} := \bigoplus_{i=0}^{\nu-1} \mathcal{L}^{(-i)}.$$

Remark 1.7. For $i > \nu$, one should note that:

$$\begin{aligned}\mathcal{L}^{(-i)} &= \mathcal{L}^{-i} \otimes \mathcal{O}_X\left(\left[\frac{iD}{\nu}\right]\right) = \mathcal{L}^{-i} \otimes \mathcal{O}_X\left(\left[\frac{(i-\nu+\nu)D}{\nu}\right]\right) \\ &= \mathcal{L}^{-i} \otimes \mathcal{O}_X(D) \otimes \mathcal{O}_X\left(\left[\frac{(i-\nu)D}{\nu}\right]\right) = \mathcal{L}^{-i+\nu} \otimes \mathcal{O}_X\left(\left[\frac{(i-\nu)D}{\nu}\right]\right) \\ &= \mathcal{L}^{(-i+\nu)}.\end{aligned}$$

We should note that for two positive integers i, j one has:

$$\left[\frac{iD}{\nu}\right] + \left[\frac{jD}{\nu}\right] \leq \left[\frac{(i+j)D}{\nu}\right]. \tag{1.2}$$

This enables to define the multiplication in $\mathcal{A}$ by:

$$\mathcal{L}^{(-i)} \times \mathcal{L}^{(-j)} \to \mathcal{L}^{-i-j} \otimes \mathcal{O}_X\left(\left[\frac{iD}{\nu}\right] + \left[\frac{jD}{\nu}\right]\right) \to \mathcal{L}^{-i-j} \otimes \mathcal{O}_X\left(\left[\frac{(i+j)D}{\nu}\right]\right) = \mathcal{L}^{(-i-j)}.$$

By Remark 1.7 for $i + j > \nu$ one has $\mathcal{L}^{(-i-j)} = \mathcal{L}^{(-i-j+\nu)}$. Therefore, $\mathcal{A}$ is an $\mathcal{O}_X$-algebra. We have the inclusion $\mathcal{L}^{-i} \hookrightarrow \mathcal{L}^{(-i)}$, which gives a morphism of $\mathcal{O}_X$-modules:

$$\mathcal{A}' \hookrightarrow \mathcal{A}, \tag{1.3}$$

and one checks immediately that it is also a morphism of $\mathcal{O}_X$-algebras.

The $\mathcal{O}_X$-algebra $\mathcal{A}'$ is a finite sum of invertible sheaves so it is coherent, or finitely presented and hence in particular it is a quasicoherent $\mathcal{O}_X$-algebra.

Definition 1.8. A ramified covering $\tau : Z \to X$ over a complex manifold X is called *a cyclic covering of degree* ν if the group $\mathrm{Aut}(Z/X)$ of automorphisms of Z preserving fibers over X is cyclic of order ν (then it follows that $\mathrm{Aut}(Z/X)$ acts transitively on fibers $Z_x := \tau^{-1}(x), x \in X$).

There is a cyclic group $G = \langle \sigma \rangle \cong \mathbb{Z}/\nu$ of order ν which acts on the $\mathcal{O}_X$-algebra $\mathcal{A}'$, via the multiplication by ζ^i of the elements of the component $\mathcal{L}^{-i}$, where ζ is a fixed primitive ν-th root of unity and $i \in \{0, 1, ..., \nu-1\}$. Then, the cyclic group $G = \langle \sigma \rangle$ acts transitively on fibers of the map $f : W' \to X$, where $W' = \mathrm{Spec}_X(\mathcal{A}')$. Hence, W' is a cyclic covering of X of degree ν. Since every automorphism of a variety extends to an automorphism of the normalization of a variety, the cyclic group G acts on the normalization W of W'. This leads to the following definition:

Definition 1.9. Let $\pi : W \to X$ be the finite morphism obtained by normalizing $f : W' \to X$. The variety W is called *the cyclic covering of degree ν obtained by taking the ν-th root out of D.*

Since W is a cyclic covering of degree ν, the cyclic group $G = \langle \sigma \rangle$ acts on $\pi_* \mathcal{O}_W$, decomposing it as a direct sum of spaces of eigenvectors corresponding to eigenvalues ζ^i, We give here the theorem which describes this decomposition and which was shown by Hélène Esnault in [19].

Theorem 1.2. *([20] §3) In the previous notations, one has the decomposition:*

$$\pi_* \mathcal{O}_W = \bigoplus_{i=0}^{\nu-1} \mathcal{L}^{(-i)},$$

in eigenspaces for the action by the cyclic group G.

Proof. Let $X_0 \subset X$ be an open subset such that

$$\operatorname{codim}_X(X \backslash X_0) \geq 2.$$

Let $W_0 = \pi^{-1}(X_0)$ be the normalization of $W_0' = \operatorname{Spec}_{X_0}(\mathcal{A}'_{|X_0})$. Let i and i' be the induced inclusions, as it is shown in the diagram below. As W is normal we have

$$i'_* \mathcal{O}_{W_0} = \mathcal{O}_W.$$

$$\begin{tikzcd} W_0 \arrow[r, "i'"'] \arrow[d, "\pi_0"'] & W \arrow[d, "\pi"'] \\ X_0 \arrow[r, "i"'] & X \end{tikzcd}$$

The diagram commutes and we have :

$$i_* \pi_{0*} \mathcal{O}_{W_0} = \pi_* i'_* \mathcal{O}_{W_0} = \pi_* \mathcal{O}_W. \tag{1.4}$$

Since $\mathcal{A} = \bigoplus_{i=0}^{\nu-1} \mathcal{L}^{(-i)}$, it is locally free and

$$i_* \mathcal{A}_{|X_0} = \mathcal{A}. \tag{1.5}$$

Moreover, let us suppose that $U \subset X_0$ is an affine set. By abusing the notation we denote $W_0 = \pi^{-1}(U)$ and $W_0' = \operatorname{Spec}(\mathcal{A}'_{|U}(U))$. The equations (1.4) and (1.5) are clearly satisfied on U. Let $A' = H^0(U, \mathcal{A}')$ and let $A = H^0(U, \mathcal{A})$. We will suppose that X_0 is such that for all affine open subsets $U \subset X_0$ one has that A is the normalization of A'. (We will prove later that such one X_0 exists.) Then Spec A is the normalization of Spec A'. In other words, Spec A is isomorphic to W_0. One gets:

$$\pi_{0*} \mathcal{O}_{W_0} = \mathcal{A}_{|U}.$$

By (1.4) and (1.5) we have: $\pi_* \mathcal{O}_W = \bigoplus_{i=0}^{\nu-1} \mathcal{L}^{(-i)}$. □

We are going to show now that we can take $X_0 = X \backslash \mathrm{Sing}(D)$ in the previous proof. Let U be an open affine subset of $X_0 = X \backslash \mathrm{Sing}(D)$ and set $A = H^0(U, \mathcal{A})$, $A' = H^0(U, \mathcal{A}')$. We have to prove that A is the normalization of A'. Let $U = \mathrm{Spec}\, B$, where $B = H^0(U, \mathcal{O}_U)$ and moreover we can and do suppose that on U the divisor D consists of one component $D = \alpha_1 D_1$. Let us fix an isomorphism $\mathcal{L}^i \cong \mathcal{O}_U$, for each $i = 0, ..., \nu - 1$. Let $f_1 \in B$ be a local equation of the component D_1. Then, for some invertible function $u \in B^*$ the section $s \in H^0(U, \mathcal{L}^\nu) \cong B$ which defines the cyclic covering is identified with $u f_1^{\alpha_1}$.
Since $\mathcal{L}^i \cong \mathcal{O}_U$, the space of sections $H^0(U, \mathcal{A}')$ is the B-algebra:

$$A' = H^0(U, \mathcal{A}') = \bigoplus_{i=0}^{\nu-1} H^0(U, \mathcal{L}^{-i}) \cong \bigoplus_{i=0}^{\nu-1} B t^i \cong B[t]/(t^\nu - u f_1^{\alpha_1}).$$

Note that $u = t^\nu f_1^{-\alpha_1}$. Also, we have the B-algebra:

$$A = H^0(U, \mathcal{A}) = \bigoplus_{i=0}^{\nu-1} B t^i f_1^{-[\frac{i\alpha_1}{\nu}]},$$

where $t^i f_1^{-[\frac{i\alpha_1}{\nu}]}$ is a generator of the space of sections of $\mathcal{L}^{(-i)}$. The multiplication of sections in A is realized via:

$$\begin{aligned} t^i f_1^{-[\frac{i\alpha_1}{\nu}]} \cdot t^j f_1^{-[\frac{j\alpha_1}{\nu}]} &= t^{i+j} f_1^{-[\frac{i\alpha_1}{\nu}]-[\frac{i\alpha_1}{\nu}]} \\ &= f_1^{-[\frac{i\alpha_1}{\nu}]-[\frac{i\alpha_1}{\nu}]+[\frac{(i+j)\alpha_1}{\nu}]} t^{i+j} f_1^{-[\frac{(i+j)\alpha_1}{\nu}]} \\ &= b_{ij} t^{i+j} f_1^{-[\frac{(i+j)\alpha_1}{\nu}]}, \end{aligned} \tag{1.6}$$

where $b_{ij} = f_1^{-[\frac{i\alpha_1}{\nu}]-[\frac{i\alpha_1}{\nu}]+[\frac{(i+j)\alpha_1}{\nu}]}$, for $i, j \in \{0, 1, ..., \nu - 1\}$. One has to note that (1.2) yields $b_{ij} \in B$. In the case when $i + j \geq \nu$, one has:

$$t^i f_1^{-[\frac{i\alpha_1}{\nu}]} \cdot t^j f_1^{-[\frac{j\alpha_1}{\nu}]} = b_{ij} u t^{i+j-\nu} f_1^{-[\frac{(i+j-\nu)\alpha_1}{\nu}]}. \tag{1.7}$$

It is obvious that the morphism (1.3) induces an inclusion:

$$A' \hookrightarrow A. \tag{1.8}$$

Let $\gcd(\nu, \alpha_1) = e, \mu = \dfrac{\nu}{e}$ and $\alpha = \dfrac{\alpha_1}{e}$. We assume that U is chosen small enough such that there exists $u_0 \in B^*$ with $u_0^e = u$, then one has:

$$t^\nu - u f_1^{\alpha_1} = \prod_{j=0}^{e-1} (t^\mu - u_0 \xi^j f_1^\alpha), \tag{1.9}$$

where ξ is a primitive e-th root of unity.

Now when we have set up this framework, let us give the proof that A is the normalization of A'.

Claim 1.3. *The normalization of A' is isomorphic to:*

$$A'_{nor} \cong \prod_{j=0}^{e-1} (A'_j)_{nor},$$

where $(A'_j)_{nor} \cong \bigoplus_{i=0}^{\mu-1} Bt^i f_1^{-[\frac{i\alpha}{\mu}]}$ *and on* $(A'_j)_{nor}$ *one has* $t^\mu f_1^{-\alpha} = u_0 \xi^j$.

Proof. By the Chinese remainder theorem and (1.9) one has:

$$A' = B[t]/(t^\nu - u f_1^{\alpha_1}) \cong \prod_{j=0}^{e-1} B[t]/(t^\mu - u_0 \xi^j f_1^\alpha).$$

We define:

$$A'_j := B[t]/(t^\mu - u_0 \xi^j f_1^\alpha).$$

One should note that on A'_j one has:

$$t^\mu f_1^{-\alpha} = u_0 \xi^j \in B^*. \tag{1.10}$$

From the geometric point of view, the previous decomposition of A' implies that Spec A' has e irreducible components given by the equations:

$$t^\mu - u_0 \xi^j f_1^\alpha = 0,$$

for $j \in \{0, ..., e-1\}$.

Now, let us fix $j \in \{0, ..., e-1\}$ and consider one of the components of A' :

$$A'_j = B[t]/(t^\mu - w f_1^\alpha) \cong \bigoplus_{i=0}^{\mu-1} Bt^i,$$

where

$$w = u_0 \xi^j.$$

Since μ and α are co-prime we can find positive integers i_0 and j_0 such that:

$$j_0 \alpha - \mu i_0 = 1.$$

One should note:

$$\left(\frac{t^{j_0}}{f_1^{i_0}}\right)^\mu = \left(\frac{t^\mu}{f_1^\alpha}\right)^{j_0} f_1 = w^{j_0} f_1, \qquad \left(\frac{t^{j_0}}{f_1^{i_0}}\right)^\alpha = w^{i_0} t,$$

since $w = \dfrac{t^\mu}{f_1^\alpha}$.

Let us define the morphism $\phi : B[t] \to B[v]$ by

$$\phi(t) = \frac{v^\alpha}{w^{i_0}}.$$

Since
$$\phi(t^{\mu} - wf_1^{\alpha}) = \frac{v^{\alpha\mu} - w^{i_0\mu+1} f_1^{\alpha}}{w^{i_0\mu}} = \frac{v^{\alpha\mu} - w^{j_0\alpha} f_1^{\alpha}}{w^{i_0\mu}},$$
the image of the ideal generated by $t^{\mu} - wf_1^{\alpha}$ by the map ϕ belongs to the ideal generated by $v^{\mu} - w^{j_0} f_1$, and one has an induced morphism:

$$B[t]/(t^{\mu} - wf_1^{\alpha}) \to B[v]/(v^{\mu} - w^{j_0} f_1).$$

We define:
$$(A_j')_{nor} := B[v]/(v^{\mu} - w^{j_0} f_1),$$
and let us prove that it is normal. We have the map

$$f : \mathrm{Spec}\,(B[t]/(t^{\mu} - wf_1^{\alpha})) \to U,$$

see Definition 1.9, and the map

$$g : \mathrm{Spec}\,\big(B[v]/(v^{\mu} - w^{j_0} f_1)\big) \to \mathrm{Spec}\,(B[t]/(t^{\mu} - wf_1^{\alpha}))\,,$$

which is induced by $\phi : B[t] \to B[v]$. For $x \in U$ we choose $f_2, ..., f_n$ such that $w^{j_0} f_1, f_2, ..., f_n$ is a local parameter system at x. Then, $v, f_2, ..., f_n$ is a local parameter system at $(f \circ g)^{-1}(x)$. Therefore, $\mathrm{Spec}\,(B[v]/(v^{\mu} - w^{j_0} f_1))$ is smooth and so it is normal. In other words, $(A_j')_{nor}$ is normal.

Let $k \in \{0, 1, ..., \mu - 1\}$ and let $k\alpha = \mu a_k + r_k$ be the Euclidean division of $k\alpha$ by μ, then:
$$a_k = \left[\frac{k\alpha}{\mu}\right].$$
One should note that:

$$t^k = \left(\frac{v^{\alpha}}{w^{i_0}}\right)^k = \frac{v^{\mu a_k + r_k}}{w^{i_0 k}} = \frac{f_1^{a_k} w^{a_k j_0} v^{r_k}}{w^{i_0 k}} = w^{a_k j_0 - i_0 k} f_1^{a_k} v^{r_k},$$

$$t^k f_1^{-a_k} = \lambda_{r_k} v^{r_k}, \lambda_{r_k} \in B^*.$$

Now we remark that when k describes the set $\{0, 1, ..., \mu - 1\}$ then r_k describes the same set. Indeed, let us reduce the equation $k\alpha = \mu a_k + r_k$ modulo μ. One gets $k\alpha \equiv r_k \pmod{\mu}$. Since, $j_0\alpha - \mu i_0 = 1$ one has that α modulo μ is a unit, so the set of values of r_k is $\{0, 1, ..., \mu - 1\}$.

Since both k and r_k describe the set $\{0, 1, ..., \mu - 1\}$, we get:

$$(A_j')_{nor} = B[v]/(v^{\mu} - w^{j_0} f_1) = \bigoplus_{i=0}^{\mu-1} Bv^i = \bigoplus_{i=0}^{\mu-1} B\lambda_i v^i = \bigoplus_{i=0}^{\mu-1} Bt^i f_1^{-[\frac{i\alpha}{\mu}]},$$

and on $(A_j')_{nor}$ one has
$$t^{\mu} f_1^{-\alpha} = u_0 \xi^j \in B^*. \tag{1.11}$$

Now, we want to prove that $(A'_j)_{nor}$ is the normalization of A'_j. In other words, we will show that $(A'_j)_{nor}$ is the integral closure of A'_j in the field of fractions $\mathrm{Frac}(A'_j)$.

Since $A'_j \cong \bigoplus_{i=0}^{\mu-1} Bt^i$ and $(A'_j)_{nor} \cong \bigoplus_{i=0}^{\mu-1} Bt^i f_1^{-[\frac{i\alpha}{\mu}]}$ one has the inclusion:

$$A'_j \hookrightarrow (A'_j)_{nor}.$$

So,

$$\mathrm{Frac}(A'_j) \subseteq \mathrm{Frac}((A'_j)_{nor}).$$

On the other hand, since $f_1 \in B$ one has:

$$(A'_j)_{nor} \subseteq \mathrm{Frac}(A'_j)$$

and

$$\mathrm{Frac}((A'_j)_{nor}) \subseteq \mathrm{Frac}(A'_j).$$

Hence,

$$\mathrm{Frac}(A'_j) \cong \mathrm{Frac}((A'_j)_{nor}).$$

Since $(A'_j)_{nor}$ is normal, it is integrally closed in $\mathrm{Frac}(A'_j)$. We denote by $\overline{A'}_j$ the integral closure of A'_j in $\mathrm{Frac}(A'_j)$. The inclusion $A'_j \hookrightarrow (A'_j)_{nor}$ implies that:

$$\overline{A'}_j \subseteq (A'_j)_{nor} \quad \text{ in } \mathrm{Frac}(A'_j).$$

Now, we will show that $(A'_j)_{nor}$ is integral over A'_j. We have to show that all generators of $(A'_j)_{nor} = \bigoplus_{i=0}^{\mu-1} Bt^i f_1^{-[\frac{i\alpha}{\mu}]}$ are roots of monic polynomials with coefficients in A'_j. Let us fix $i \in \{0, 1, ..., \mu - 1\}$ and set:

$$x_i = t^i f_1^{-[\frac{i\alpha}{\mu}]}.$$

Let us note that:

$$x_i^\mu = (t^i f_1^{-[\frac{i\alpha}{\mu}]})^\mu = (t^\mu f_1^{-\alpha})^i f_1^{-\mu[\frac{i\alpha}{\mu}]+i\alpha},$$

and $b_\mu = f_1^{-\mu[\frac{i\alpha}{\mu}]+i\alpha} \in B$.

Now, let $p_i(x) = x^\mu - b_\mu(t^\mu f_1^{-\alpha})^i$. One has that $b_\mu(t^\mu f_1^{-\alpha})^i = b_\mu(u_0\xi^j)^i \in B$, hence $p_i(x)$ is a polynomial with coefficients in A'_j. It is obvious that x_i is a root of the polynomial $p_i(x)$. Hence, $(A'_j)_{nor}$ is integral over A'_j. In other words, $(A'_j)_{nor}$ is in the integral closure of A'_j in $\mathrm{Frac}(A'_j)$. Hence, $(A'_j)_{nor} \subseteq \overline{A'}_j$ and

$$(A'_j)_{nor} = \overline{A'}_j.$$

Therefore, $(A'_j)_{nor}$ is the normalization of A'_j. Finally,

$$A'_{nor} = \prod_{j=0}^{e-1} (A'_j)_{nor} = \prod_{j=0}^{e-1} B[v]/(v^\mu - u_0^{j_0}\xi^{j_0 j} f_1) \cong \prod_{j=0}^{e-1} \bigoplus_{i=0}^{\mu-1} Bt^i f_1^{-[\frac{i\alpha}{\mu}]}.$$

$\square$

Claim 1.4. *In the previous notations, $A \cong A'_{nor}$.*

Proof. We have:

$$A = \bigoplus_{i=0}^{\nu-1} Bt^i f_1^{-[\frac{i\alpha_1}{\nu}]} \tag{1.12}$$

and

$$A'_{nor} = \prod_{j=0}^{e-1} (A'_j)_{nor} \cong \prod_{j=0}^{e-1} \bigoplus_{i=0}^{\mu-1} Bt^i f_1^{-[\frac{i\alpha}{\mu}]}.$$

Let us define a map $\varphi : A \to A'_{nor}$ by:

$$\varphi\left(t^i f_1^{-[\frac{i\alpha_1}{\nu}]}\right) = \left(u_0^k t^p f_1^{-[\frac{p\alpha}{\mu}]}, u_0^k \xi^k t^p f_1^{-[\frac{p\alpha}{\mu}]}, u_0^k \xi^{2k} t^p f_1^{-[\frac{p\alpha}{\mu}]} ..., u_0^k \xi^{(e-1)k} t^p f_1^{-[\frac{p\alpha}{\mu}]}\right),$$

where the right hand side is an e-tuple and $i = k\mu + p$ is the Euclidean division of i by μ. Then, $k \in \{0, ..., e-1\}$ and $p \in \{0, 1, ..., \mu - 1\}$.

Let us check that φ is a well defined morphism of algebras, i.e. that for all $i, j \in \{0, ..., \nu - 1\}$ one has:

$$\varphi\left(t^i f_1^{-[\frac{i\alpha_1}{\nu}]} \cdot t^j f_1^{-[\frac{j\alpha_1}{\nu}]}\right) = \varphi\left(t^i f_1^{-[\frac{i\alpha_1}{\nu}]}\right) \cdot \varphi\left(t^j f_1^{-[\frac{j\alpha_1}{\nu}]}\right). \tag{1.13}$$

Let us fix notations which will be used in the rest of the proof. Let $i = k\mu+p$ and $j = l\mu + q$ be the Euclidean divisions of i and j by μ. We will distinguish two cases. The first one for $i + j < \nu$ and the second for $i + j \geq \nu$.

1. Let us suppose that $i+j < \nu$ and $i+j = m\mu+r$ is the Euclidean division of $i + j$ by μ. Then by (1.6) one gets:

$$\begin{aligned}\varphi\left(t^i f_1^{-[\frac{i\alpha_1}{\nu}]} \cdot t^j f_1^{-[\frac{j\alpha_1}{\nu}]}\right) &= \varphi\left(b_{ij} t^{i+j} f_1^{-[\frac{(i+j)\alpha_1}{\nu}]}\right) \\ &= \left(b_{ij} u_0^m t^r f_1^{-[\frac{r\alpha}{\mu}]}, ..., b_{ij} u_0^m \xi^{m(e-1)} t^r f_1^{-[\frac{r\alpha}{\mu}]}\right).\end{aligned}$$

   (a) Let us suppose that $p + q < \mu$, then one gets that $k + l = m$ and $p + q = r$. Hence,

$$\begin{aligned}&\varphi\left(t^i f_1^{-[\frac{i\alpha_1}{\nu}]}\right) \cdot \varphi\left(t^j f_1^{-[\frac{j\alpha_1}{\nu}]}\right) \\ &= \left(u_0^k t^p f_1^{-[\frac{p\alpha}{\mu}]}, ..., u_0^k \xi^{k(e-1)} t^p f_1^{-[\frac{p\alpha}{\mu}]}\right) \cdot \left(u_0^l t^q f_1^{-[\frac{q\alpha}{\mu}]}, ..., u_0^l \xi^{l(e-1)} t^q f_1^{-[\frac{q\alpha}{\mu}]}\right) \\ &= \left(u_0^m t^r f_1^{-[\frac{p\alpha}{\mu}]-[\frac{q\alpha}{\mu}]}, ..., u_0^m \xi^{m(e-1)} t^r f_1^{-[\frac{p\alpha}{\mu}]-[\frac{q\alpha}{\mu}]}\right).\end{aligned}$$

   From the equation (1.6) one gets:

$$f_1^{-[\frac{(k\mu+p)\alpha}{\mu}]-[\frac{(lm+q)\alpha}{\mu}]} = b_{ij} f_1^{-[\frac{(m\mu+r)\alpha}{\mu}]}$$

which yields:

$$f_1^{-[\frac{p\alpha}{\mu}]-[\frac{q\alpha}{\mu}]} = b_{ij} f_1^{-[\frac{r\alpha}{\mu}]}.$$

So, we get that the equality (1.13) holds in this case.

(b) Now, if $p+q \geq \mu$ then one has $p+q = \mu + r_1$ where $r_1 < \mu$ and $i+j = (k+l+1)\mu + r_1$, which implies that $k+l+1 = m$ and $r_1 = r$. Then one has:

$$\begin{aligned}
&\varphi\left(t^i f_1^{-[\frac{i\alpha_1}{\nu}]}\right) \cdot \varphi\left(t^j f_1^{-[\frac{j\alpha_1}{\nu}]}\right) \\
&= \left(u_0^k t^p f_1^{-[\frac{p\alpha}{\mu}]}, ..., u_0^k \xi^{k(e-1)} t^p f_1^{-[\frac{p\alpha}{\mu}]}\right) \cdot \left(u_0^l t^q f_1^{-[\frac{q\alpha}{\mu}]}, ..., u_0^l \xi^{l(e-1)} t^q f_1^{-[\frac{q\alpha}{\mu}]}\right) \\
&= \left(u_0^{m-1} t^r t^\mu f_1^{-[\frac{p\alpha}{\mu}]-[\frac{q\alpha}{\mu}]}, ..., u_0^{m-1} \xi^{(m-1)(e-1)} t^r t^\mu f_1^{-[\frac{p\alpha}{\mu}]-[\frac{q\alpha}{\mu}]}\right).
\end{aligned}$$

The equation (1.6) implies that:

$$f_1^{-[\frac{p\alpha}{\mu}]-[\frac{q\alpha}{\mu}]} = b_{ij} f_1^{-\alpha} f_1^{-[\frac{r\alpha}{\mu}]}.$$

Now, using the fact that on $(A'_s)_{nor}$ for $s \in \{0, ..., e-1\}$ one has

$$t^\mu f_1^{-\alpha} = u_0 \xi^s,$$

we get:

$$\begin{aligned}
&\varphi\left(t^i f_1^{-[\frac{i\alpha_1}{\nu}]}\right) \cdot \varphi\left(t^j f_1^{-[\frac{j\alpha_1}{\nu}]}\right) \\
&= \left(u_0^{m-1} t^r t^\mu f_1^{-[\frac{p\alpha}{\mu}]-[\frac{q\alpha}{\mu}]}, ..., u_0^{m-1} \xi^{(m-1)(e-1)} t^r t^\mu f_1^{-[\frac{p\alpha}{\mu}]-[\frac{q\alpha}{\mu}]}\right) \\
&= \left(u_0^{m-1} t^r t^\mu b_{ij} f_1^{-\alpha} f_1^{-[\frac{r\alpha}{\mu}]}, ..., u_0^{m-1} \xi^{(m-1)(e-1)} t^r t^\mu b_{ij} f_1^{-\alpha} f_1^{-[\frac{r\alpha}{\mu}]}\right) \\
&= \left(b_{ij} u_0^m t^r f_1^{-[\frac{r\alpha}{\mu}]}, ..., b_{ij} u_0^m \xi^{m(e-1)} t^r f_1^{-[\frac{r\alpha}{\mu}]}\right).
\end{aligned}$$

2. Let us now suppose that $i+j \geq \nu$. Let $i+j-\nu = c\mu + d$ be the Euclidean divisions of $i+j-\nu$ by μ. We have $i+j = (c+e)\mu + d$. By (1.7) and (1.11) one has:

$$\begin{aligned}
\varphi\left(t^i f_1^{-[\frac{i\alpha_1}{\nu}]} \cdot t^j f_1^{-[\frac{j\alpha_1}{\nu}]}\right) &= \varphi\left(b_{ij} u t^{i+j-\nu} f_1^{-[\frac{(i+j-\nu)\alpha_1}{\nu}]}\right) \\
&= \varphi\left(b_{ij} u t^{c\mu+d} f_1^{-[\frac{(c\mu+d)\alpha}{\mu}]}\right) \\
&= \left(b_{ij} u u_0^c t^d f_1^{-[\frac{d\alpha}{\mu}]}, ..., b_{ij} u u_0^c \xi^{c(e-1)} t^d f_1^{-[\frac{d\alpha}{\mu}]}\right)
\end{aligned}$$

We will again distinguish two cases:

(a) The first case is the one where $p+q<\mu$. Then from $i+j=(k+l)\mu+p+q$ one gets $k+l=c+e$ and $d=p+q$. Also, using (1.7) one gets:
$$f_1^{-[\frac{p\alpha}{\mu}]-[\frac{q\alpha}{\mu}]}=b_{ij}f_1^{-[\frac{d\alpha}{\mu}]}.$$
Therefore,
$$\begin{aligned}&\varphi\left(t^i f_1^{-[\frac{i\alpha_1}{\nu}]}\right)\cdot\varphi\left(t^j f_1^{-[\frac{j\alpha_1}{\nu}]}\right)\\&=\left(u_0^k t^p f_1^{-[\frac{p\alpha}{\mu}]},...,u_0^k\xi^{k(e-1)}t^p f_1^{-[\frac{p\alpha}{\mu}]}\right)\cdot\left(u_0^l t^q f_1^{-[\frac{q\alpha}{\mu}]},...,u_0^l\xi^{l(e-1)}t^q f_1^{-[\frac{q\alpha}{\mu}]}\right)\\&=\left(u_0^{c+e}t^d f_1^{-[\frac{p\alpha}{\mu}]-[\frac{q\alpha}{\mu}]},...,u_0^{c+e}\xi^{(c+e)(e-1)}t^d f_1^{-[\frac{p\alpha}{\mu}]-[\frac{q\alpha}{\mu}]}\right)\\&=\left(b_{ij}uu_0^c t^d f_1^{-[\frac{d\alpha}{\mu}]},...,b_{ij}uu_0^c\xi^{c(e-1)}t^d f_1^{-[\frac{d\alpha}{\mu}]}\right)\end{aligned}$$
where the last equality holds since $u_0^e=u$ and $\xi^e=1$.

(b) For $p+q\geq\mu$ we take $p+q=\mu+r_1$, where $r_1<\mu$, and we get $i+j=(k+l+1)\mu+r_1$. Hence, $c+e=k+l+1$ and $d=r_1$. Using (1.7) one has:
$$f_1^{-[\frac{p\alpha}{\mu}]-[\frac{q\alpha}{\mu}]}=b_{ij}f_1^{-\alpha}f_1^{-[\frac{d\alpha}{\mu}]}.$$
Repeating the same arguments as before one gets that the equality (1.13) holds.

Therefore, the map φ is a well defined morphism of B-algebras between A and A'_{nor}. Let us show that φ is a surjective morphism. In order to do that we will show that for every element which generates the B-algebra A'_{nor}, i.e. $\left(0,...,t^j f_1^{-[\frac{j\alpha}{\mu}]},...,0\right)$ $j\in\{0,1,...,\mu-1\}$, there are elements $a_i\in B$ such that:
$$\varphi\left(\sum_{i=0}^{\nu-1}a_i t^i f_1^{-[\frac{i\alpha_1}{\nu}]}\right)=\left(0,...,t^j f_1^{-[\frac{j\alpha}{\mu}]},...,0\right),\tag{1.14}$$
where the vector on the right-hand side has non-zero coordinate only on d-th place, where $d\in\{0,...,e-1\}$. We suppose that d and j are fixed. One should note that for $i\in\{0,1,...,\nu-1\}$:
$$\begin{aligned}&\varphi\left(\sum a_i t^i f_1^{-[\frac{i\alpha_1}{\nu}]}\right)\\&=\left(\sum a_i u_0^{s_i}t^{j_i}f_1^{-[\frac{j_i\alpha}{\mu}]},\sum a_i u_0^{s_i}\xi^{s_i}t^{j_i}f_1^{-[\frac{j_i\alpha}{\mu}]},...,\sum a_i u_0^{s_i}\xi^{(e-1)s_i}t^{j_i}f_1^{-[\frac{j_i\alpha}{\mu}]}\right),\end{aligned}\tag{1.15}$$
where $i=s_i\mu+j_i$ is the Euclidean division of i by μ. Comparing this equality to the equality (1.14) one gets μ different systems of equations:

- for $k \neq j$ and $k \in \{0, 1, ..., \mu - 1\}$ we have systems:
$$M\ {}^t(a_k, a_{\mu+k}, ..., a_{(e-1)\mu+k}) = (0, ..., 0, ...0); \tag{1.16}$$
- otherwise, we have the system:
$$M\ {}^t(a_j, a_{\mu+j}, ..., a_{(e-1)\mu+j}) = (0, ..., 1, ...0), \tag{1.17}$$
where the right side vector has the only non-zero coordinate on d-th place.

The matrix M is:
$$\begin{pmatrix} 1 & u_0 & u_0^2 & \dots & u_0^{e-1} \\ 1 & u_0\xi & u_0^2\xi^2 & \dots & u_0^{e-1}\xi^{e-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & u_0\xi^d & u_0^2\xi^{2d} & \dots & u_0^{e-1}\xi^{(e-1)d} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & u_0\xi^{e-1} & u_0^2\xi^{2(e-1)} & \dots & u_0^{e-1}\xi^{(e-1)(e-1)} \end{pmatrix}.$$
and
$$\begin{aligned} \det M &= u_0^{\frac{e(e-1)}{2}} \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \xi & \xi^2 & \dots & \xi^{e-1} \\ \vdots & \vdots & & \vdots & \\ 1 & \xi^d & \xi^{2d)} & \dots & \xi^{(e-1)d} \\ \vdots & \vdots & & \vdots & \\ 1 & \xi^{e-1} & \xi^{2(e-1)} & \dots & \xi^{(e-1)(e-1)} \end{vmatrix} \\ &= u_0^{\frac{e(e-1)}{2}} \prod_{0 \leq k < l \leq e-1} (\xi^l - \xi^k), \end{aligned}$$
where the last equality is the computation of the Vandermonde determinant. Obviously, $\det M$ is invertible, therefore the system (1.17) has a unique solution. Hence, the morphism φ is surjective. Repeating the same arguments one can prove that $\operatorname{Ker} \varphi = \{0\}$. Therefore, the morphism φ is an isomorphism between the finitely generated B-algebras A and A'_{nor}.

The last step that needs to be proved is that the decomposition (1.12) is the decomposition of $\pi_*\mathcal{O}_W(U)$ into isotypical components for the action by the cyclic automorphism group $G = \langle\sigma\rangle$ of W.

Let $(x_1, x_2, ..., x_n)$ be a local system of coordinates on U. Let U be small enough, then $\pi^{-1}(U) \cong U \times G$. Then the action of G on $\pi^{-1}(U)$ is given by:
$$\sigma.(x_1, x_2, ..., x_n, g) = (x_1, x_2, ..., x_n, \sigma g),$$
where $g \in G$. Therefore, the action of G on $\mathcal{O}_W(\pi^{-1}(U))$ is given by:
$$\sigma.f = \sum_{i=0}^{\nu-1} b_i(\zeta t)^i f_1^{-[\frac{i\alpha_1}{\nu}]},$$

for a holomorphic function $f = \sum_{i=0}^{\nu-1} b_i t^i f_1^{-[\frac{i\alpha_1}{\nu}]} \in \mathcal{O}_W(\pi^{-1}(U))$. The eigenvalues of this action are $\{1, \zeta, \zeta^2, ..., \zeta^{\nu-1}\}$ and an easy calculation shows that the space of eigenvectors for the eigenvalue ζ^j is generated by the vector $t^j f_1^{-[\frac{j\alpha_1}{\nu}]}$, i.e. $\mathcal{L}^{(-j)}$ are the isotypical components for the action by the cyclic automorphism group G. $\square$

Remark 1.10. Since A is the normalization of A', one gets that $\mathrm{Spec}_X \mathcal{A}$ is the normalization of $W' = \mathrm{Spec}_X \mathcal{A}'$. Then by Definition 1.9 one gets that W is isomorphic to $\mathrm{Spec}_X \mathcal{A}$.

Lemma 1.5. *In the previous notations, let $\Gamma = \sum_{\frac{\alpha_i}{\nu} \notin \mathbb{Z}} D_i$. The divisor Γ is the support of the branch locus of the cyclic covering W.*

Proof. We recall that $D = \sum_{j=1}^{k} \alpha_j D_j$. Let $\tilde{D} = D - \nu \hat{D}$, where $\hat{D} = \sum_{\alpha_j \geq \nu} D_j$. Let $\mathcal{L}' = \mathcal{L}(-\hat{D})$, then we have:

$$\mathcal{L}'^{\nu} = \mathcal{L}^{\nu}(-\nu\hat{D}) = \mathcal{O}_X(D) \otimes \mathcal{O}_X(-\nu\hat{D}) = \mathcal{O}_X(\tilde{D}).$$

Also, we have:

$$\begin{aligned}
\mathcal{L}'^{(-i)} := (\mathcal{L}')^{-i} \otimes \mathcal{O}_X\left(\left[\frac{i\tilde{D}}{\nu}\right]\right) &= \mathcal{L}^{-i} \otimes \mathcal{O}_X(i\hat{D}) \otimes \mathcal{O}_X\left(\left[\frac{iD - i\nu\hat{D}}{\nu}\right]\right) \\
&= \mathcal{L}^{-i} \otimes \mathcal{O}_X(-i\hat{D} \otimes i\hat{D}) \otimes \mathcal{O}_X\left(\left[\frac{iD}{\nu}\right]\right) \\
&= \mathcal{L}^{-i} \otimes \mathcal{O}_X\left(\left[\frac{iD}{\nu}\right]\right) = \mathcal{L}^{(-i)}.
\end{aligned}$$

Then, $W \cong \mathrm{Spec}_X \bigoplus_{i=0}^{\nu-1} \mathcal{L}^{(-i)} \cong \mathrm{Spec}_X \bigoplus_{i=0}^{\nu-1} \mathcal{L}'^{(-i)}$. We can repeat this process until we get an effective divisor $\tilde{D}$ whose coefficients are less than ν. The divisor $\tilde{D}$ is a divisor whose support is Γ, since $\tilde{D} = \sum_{j=1}^{m} \beta_j D_j$, where $0 < \beta_j \leq \nu - 1$ are the remainders of the Euclidean division of α_j by ν and $m \leq k$, up to reordering the D_j. So, the branch locus is contained in the support of the divisor Γ.

We should prove now that the cyclic covering W is ramified over every component of $\tilde{D}$. Locally, on some small open set $U \subset X \backslash \mathrm{Sing}(\tilde{D})$ we can suppose that $\tilde{D}$ contains just one component $\beta_j D_j, j \in \{1, ..., m\}$. Let f_j be a local equation of D_j. We suppose that $\gcd(\nu, \beta_j) = e_j$. We set $\mu_j = \dfrac{\nu}{e_j}$. Then, as we saw in the course of the proof of Claim 1.3, after normalization, the cyclic covering will locally be:

$$\mathrm{Spec}\left(\prod_{k=0}^{e_j-1} B[v]/(v^{\mu_j} - w_k f_j)\right),$$

where $w_k \in B^*$. Hence, it is obvious that the cyclic covering will not be étale over D_j, for $j \in \{1, ..., m\}$. Therefore, the cyclic covering is ramified over all the components of $\tilde{D}$ and Γ is the support of the branch locus of the cyclic covering.

□

Remark 1.11. In the previous notation, the cyclic covering is not ramified if all the coefficients α_j are divisible by ν.

Proposition 1.6. *The cyclic covering $\pi : W \to X$ is étale over $X - \Gamma$ and W is non-singular over $X - Sing(\Gamma)$.*

Proof. This is a direct consequence of Claim 1.3. □

Definition 1.12. Let X be a complex manifold of dimension n. A reduced divisor $D = \sum_i D_i$ on X is a normal crossings divisor if each component D_i is smooth and D is defined in a neighborhood of any point by an equation of the type:

$$z_1 z_2 ... z_k = 0, k \leq n,$$

where $z_1, z_2, ..., z_n$ is a local coordinate system on X. A non-reduced divisor $D = \sum_i \alpha_i D_i, \alpha_i \neq 0$ on X is a normal crossings divisor if $D_{\text{red}} := \sum_i D_i$ is a normal crossings divisor.

Remark 1.13. In the literature, the definition given above sometimes corresponds to the definition of a simple normal crossing divisor.

Remark 1.14. In the notations of Claim 1.3, let us assume that D (the branch divisor of the cyclic covering) is a smooth divisor on the open set U and let D' be a divisor on U such that $D + D'$ is a normal crossing divisor. Then, following the proof of Claim 1.3 one gets that $\pi^*(D + D')$ is a normal crossing divisor on $\pi^*(U)$.

Now, we note that when the branch divisor of a cyclic covering is a normal crossing divisor then the singularities on the cyclic covering are not too "bad", namely they are rational singularities.

Definition 1.15. ([76] Lemma 1) A variety W has rational singularities if the minimal (or any) desingularization $\mu : \tilde{W} \to W$ satisfies $\mu_* \mathcal{O}_{\tilde{W}} = \mathcal{O}_W$.

Lemma 1.7. *([76] Lemma 2) Let X be a projective complex manifold. Let $\pi : W \to X$ be a finite morphism, where W is a normal complex projective variety. Let us suppose that the branch locus of the map $\pi : W \to X$ is a normal crossing divisor, then W has only rational singularities.*

Remark 1.16. We can construct cyclic coverings of complex manifolds in a more geometric way. This construction can be done as follows:
Let X be a connected complex manifold, $\nu \geq 1$ a fixed integer and D an effective divisor on X. Let $\mathcal{L}$ be a line bundle on X such that $\mathcal{L}^\nu \simeq \mathcal{O}_X(D)$ and $s \in \Gamma(X, \mathcal{O}_X(D))$ a section such that $\operatorname{div}(s) = D$. If $D = 0$, then s is a constant function.

Let L be the total space of the line bundle $\mathcal{L}$ and $p : L \to X$ be the bundle projection. Then, for some point $l \in L$ we have $p(l) = x \in X$ and the fiber of the pull-back line bundle $p^*\mathcal{L}$ at the point l is $(p^*\mathcal{L})_l = \mathcal{L}_x$. This implies that $(l, l) \in (p^*\mathcal{L})_l$. Then, the tautological section t of the bundle $p^*\mathcal{L}$ is defined as $t(l) = (l, l)$. The zero divisor of the section $p^*s - t^\nu$ defines an analytic subspace W of L, whose normalization is the degree-ν cyclic covering of X defined by the section s.

1.2 Logarithmic differential forms

In this section we will define several objects which will form the basis of technical tools used in the chapters that come. We will define the sheaf of logarithmic differential forms on a manifold, then families of varieties over a curve and at the end the sheaf of relative logarithmic forms of a family of varieties.

Definition 1.17. ([14],p.72) Let X be a complex manifold of dimension n and $D = \sum_{j=1}^{k} \alpha_j D_j$ be a normal crossing divisor on X. Let $X_0 = X \backslash \operatorname{Supp}(D)$ and let $j : X_0 \to X$ be the inclusion of X_0 into X. For U an open subset of X, we define :

$$\Gamma(U, \Omega^p_X(\log D_{\text{red}})) = \{\alpha \in j_*\Omega^p_{X_0} | \alpha \text{ and } d\alpha \text{ have simple poles along } D\}.$$

The sheaf $\Omega^p_X(\log D_{\text{red}})$ is the sheaf of p-differential forms with logarithmic poles along the divisor D. For simplicity we write $\Omega^p_X(\log D)$ instead of $\Omega^p_X(\log D_{\text{red}})$.

Properties:
1. $\Omega^p_X(\log D) = \wedge^p \Omega^1_X(\log D)$;

2. $\Omega^p_X(\log D)$ is a locally free sheaf:
For $x \in X$, such that $x \in D_j$ for $j = 1, 2, ..., r$ and $x \notin D_j$ for $j = r + 1, ..., k$, let $z_1, ...z_n$ be local parameters at x such that D_j is defined as $z_j = 0$ for $j = 1, 2, ..., r$.
Let:

$$\delta_j = \begin{cases} \dfrac{dz_j}{z_j}, & j = 1, ..., r; \\ dz_j, & j = r + 1, ..., n. \end{cases}$$

Then, the sheaf $\Omega^p_X(\log D)$ has a free system of generators $\{\delta_I, \#I = p\}$ given as:

$$\delta_I = \delta_{j_1} \wedge ... \wedge \delta_{j_p},$$

for $I = \{j_1, ...j_p\} \subset \{1, 2, ..., n\}$ with $j_1 < ... < j_p$.

3. $\Omega^n_X(\log D) = \omega_X \otimes \mathcal{O}_X(D)$, where ω_X is the canonical sheaf of X :
Locally, let $D = \sum_{j=1}^r D_j$ for $1 \leq r \leq n$ and in local coordinates we can suppose that $D_j = \{z_j = 0\}$. Then, as we explained before the sheaf $\Omega^p_X(\log D)$ of p-differential forms with logarithmic poles along D is locally generated by wedges of p elements in

$$\frac{dz_1}{z_1}, \frac{dz_2}{z_2}, ..., \frac{dz_r}{z_r}, dz_{r+1}, ..., dz_n.$$

In particular $\Omega^n_X(\log D)$ will be generated by

$$\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge ... \wedge \frac{dz_r}{z_r} \wedge dz_{r+1} ... \wedge dz_n = \frac{1}{z_1 ... z_r} dz_1 \wedge ... \wedge dz_r \wedge ... \wedge dz_n.$$

Since, locally $dz_1 \wedge ... \wedge dz_k \wedge ... \wedge dz_n$ generates ω_X and $\frac{1}{z_1 ... z_r}$ generates $\mathcal{O}_X(D)$, the equality holds.

Lemma 1.8. *Let Y be a smooth projective curve and let $\varphi : Y' \to Y$ be a smooth ramified covering of degree ν. Let S be the set of branch points on Y, possibly empty, i.e. φ may be an étale covering.*

1. *If $T \supset S$ is a finite set of points on Y containing S, then we have:*

$$\Omega^1_{Y'}(\log \varphi^* T) = \varphi^* \Omega^1_Y(\log T);$$

2. *If $T \subset S$, then:*

$$\varphi^* \Omega^1_Y(\log T) \subset \Omega^1_{Y'}(\log \varphi^* T).$$

Proof. 1. By the Hurwitz formula we have:

$$\begin{aligned}
\Omega^1_{Y'}(\log \varphi^* T) &= \Omega^1_{Y'} \otimes \mathcal{O}_{Y'}((\varphi^* T)_{\mathrm{red}}) \\
&= \varphi^*\Omega^1_Y \otimes \mathcal{O}_{Y'}\left(\sum_{\varphi(p') \in S} (e_{p'} - 1)p'\right) \otimes \mathcal{O}_{Y'}\left((\varphi^* T)_{\mathrm{red}}\right) \\
&= \varphi^*\Omega^1_Y \otimes \mathcal{O}_{Y'}\left(\sum_{\varphi(p') \in S} e_{p'}p'\right) \otimes \mathcal{O}_{Y'}\left(\sum_{\varphi(p') \in T\setminus S} p'\right) \\
&= \varphi^*\Omega^1_Y \otimes \varphi^*\mathcal{O}_Y(S) \otimes \mathcal{O}_{Y'}\left(\sum_{\varphi(p') \in T\setminus S} p'\right) \\
&= \varphi^*\Omega^1_Y(\log S) \otimes \varphi^*\left(\mathcal{O}_Y\left(\sum_{p \in T\setminus S} p\right)\right) \\
&= \varphi^*\Omega^1_Y(\log T),
\end{aligned}$$

where $e_{p'}$ are ramification indices at the points of Y' which lie over S. The ramification indices at the points of Y' which lie over $T\setminus S$ are equal to 1 which implies the last equalities.

2. In the same fashion as above, we get:

$$\begin{aligned}
\Omega^1_{Y'}(\log \varphi^* T) &= \varphi^*\Omega^1_Y \otimes \mathcal{O}_{Y'}\left(\sum_{\varphi(p') \in T} e_{p'}p'\right) \otimes \mathcal{O}_{Y'}\left(\sum_{\varphi(p') \in S\setminus T} (e_{p'} - 1)p'\right) \\
&= \varphi^*\Omega^1_Y(\log T) \otimes \mathcal{O}_{Y'}\left(\sum_{\varphi(p') \in S\setminus T} (e_{p'} - 1)p'\right).
\end{aligned}$$

□

Definition 1.18. Let X be a manifold of dimension $n+1$ and Y a smooth curve. Let $f : X \to Y$ be a proper surjective map whose fibers $X_y = f^{-1}(y), y \in Y$, are connected n-varieties, then the map f is called a family of n-varieties. The curve Y is called the base of the family $f : X \to Y$. If X and Y are both projective, then $f : X \to Y$ is called a projective family of n-varieties.

Definition 1.19. If for some $x \in X_y, y \in Y$ the map df_x fails to be surjective then the fiber X_y is called a singular fiber of the family $f : X \to Y$. Otherwise X_y is a smooth fiber. If all fibers in the family are smooth then we say that the family $f : X \to Y$ is smooth, or it is a family of manifolds. The set of all points $y \in Y$ such that the fiber X_y is a singular fiber, is called the discriminant locus of the family $f : X \to Y$. The components of a divisor D on X, whose support lies in fibers of the family, are called the vertical components of the divisor D. Those components of D which map onto the basis are called the horizontal components of D.

We have to underline the fact that a big part of the theory stated here works also in the case when the base of the family has a bigger dimension than 1. Here and through the rest of the thesis, we stick to the case when the base of a family is a curve.

Definition 1.20. Associated with a smooth family $f : X \to Y$, the sheaf of relative differential forms on X, denoted $\Omega^1_{X/Y}$ is defined by the exact sequence:

$$0 \to f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0.$$

The sheaf of p-relative differential forms is defined as:

$$\Omega^p_{X/Y} := \wedge^p \Omega^1_{X/Y}.$$

Definition 1.21. Differentiating along the fibers gives a differential $d : \Omega^p_{X/Y} \to \Omega^{p+1}_{X/Y}$ and the complex $(\Omega^\bullet_{X/Y}, d)$ is called the relative de Rham complex. The relative canonical bundle of the smooth family $f : X \to Y$ is defined as:

$$\omega_{X/Y} := \omega_X \otimes f^*\omega_Y^{-1},$$

where ω_X and ω_Y are the canonical sheaves of X and Y.

Now we give a more algebraic point of view on the sheaf of relative p-differential forms. The main reference for this part is a paper of Katz [40] §1.3 and §4.1.3 from [82].

Let $0 \to \mathcal{G} \to \mathcal{H} \to \mathcal{F} \to 0$ be an exact sequence of locally free sheaves on a complex manifold X. Let $\bigwedge^\bullet \mathcal{H}$ and $\bigwedge^\bullet \mathcal{H}[-m]$ be the exterior algebras of a locally free sheaf $\mathcal{H}$, where the second exterior algebra is $-m$ places shifted with respect to the first.

The Koszul filtration of $\bigwedge^\bullet \mathcal{H}$ is defined as:

$$F^m \bigwedge^{\bullet} \mathcal{H} = \operatorname{Im}\left(\bigwedge^{m} \mathcal{G} \otimes \bigwedge^{\bullet} \mathcal{H}[-m] \to \bigwedge^{\bullet} \mathcal{H} \right),$$

where in the p-th level one has:

$$F^m \bigwedge^{p} \mathcal{H} = \bigwedge^{m} \mathcal{G} \wedge \bigwedge^{p-m} \mathcal{H}.$$

The graded elements of the filtration are defined:

$$\operatorname{Gr}^m(\bigwedge^{\bullet} \mathcal{H}) = F^m/F^{m+1}(\bigwedge^{\bullet} \mathcal{H}) = \bigwedge^{m} \mathcal{G} \otimes \bigwedge^{\bullet} \mathcal{F}[-m],$$

and in the p-th level one has:

$$\mathrm{Gr}^m(\bigwedge^p \mathcal{H}) = F^m/F^{m+1}(\bigwedge^p \mathcal{H}) = \bigwedge^m \mathcal{G} \otimes \bigwedge^{p-m} \mathcal{F}.$$

We have the following exact sequence associated to this filtration:

$$0 \to \mathrm{Gr}^1 \to F^0/F^2 \to \mathrm{Gr}^0 \to 0,$$

or

$$0 \to \mathcal{G} \otimes \bigwedge^{\bullet} \mathcal{F}[-1] \to F^0/F^2(\bigwedge^{\bullet} \mathcal{H}) \to \bigwedge^{\bullet} \mathcal{F} \to 0. \tag{1.18}$$

Then in p-th level of the exact sequence (1.18) we get:

$$0 \to \mathcal{G} \otimes \wedge^{p-1}\mathcal{F} \to (F^0/F^2)^p \to \wedge^p \mathcal{F} \to 0. \tag{1.19}$$

Let $f : X \to Y$ be a smooth family of varieties (equivalently, a family of manifolds) over a curve Y. Let us apply the above formalism to the exact sequence:

$$0 \to f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0.$$

The sequence (1.18) is now:

$$0 \to f^*\Omega^1_Y \otimes \bigwedge^{\bullet} \Omega^1_{X/Y}[-1] \to F^0/F^2(\Omega^{\bullet}_X) \to \bigwedge^{\bullet} \Omega^1_{X/Y} \to 0. \tag{1.20}$$

One can easily see that $(F^0/F^2)^p = \Omega^p_X/(\wedge^2 f^*\Omega^1_Y \otimes \Omega^{p-2}_{X/Y})$. Since Y is a curve, one gets $(F^0/F^2)^p = \Omega^p_X$ and finally we get the exact sequence:

$$0 \to f^*\Omega^1_Y \otimes \Omega^{p-1}_{X/Y} \to \Omega^p_X \to \Omega^p_{X/Y} \to 0. \tag{1.21}$$

Remark 1.22. From now on, we will use the sequence (1.21) in order to define the sheaf $\Omega^p_{X/Y}$.

Let us consider the exact sequence:

$$0 \to f^*\Omega^1_Y \otimes \Omega^n_{X/Y} \to \Omega^{n+1}_X \to \Omega^{n+1}_{X/Y} \to 0.$$

Using the fact that the relative dimension of the family is n one gets:

$$\Omega^{n+1}_{X/Y} = 0.$$

Therefore,

$$\Omega^n_{X/Y} = \omega_X \otimes f^*\omega_Y^{-1} = \omega_{X/Y}.$$

The sheaf of relative differential forms can be defined also for non-smooth families. Let $f : X \to Y$ be a family of n-varieties with discriminant locus a finite set $S \subset Y$ and we will suppose that $\Delta = (f^*S)_{\mathrm{red}}$ is a normal crossing divisor. We define the sheaf of relative logarithmic 1-forms of the family $f : X \to Y$ by the sequence:

$$0 \to f^*\Omega^1_Y(\log S) \to \Omega^1_X(\log \Delta) \to \Omega^1_{X/Y}(\log \Delta) \to 0. \tag{1.22}$$

Definition 1.23. Let $f : X \to Y$ be a family of n-varieties over a curve Y with discriminant locus S. We assume that $\Delta = (f^*S)_{\mathrm{red}}$ is a normal crossing divisor. For $p \leq n$, the sheaf $\Omega^p_{X/Y}(\log \Delta)$ of relative logarithmic p-differential forms of the family is defined as

$$\Omega^p_{X/Y}(\log \Delta) := \wedge^p \Omega^1_{X/Y}(\log \Delta).$$

The relative canonical bundle of the family is defined as:

$$\omega_{X/Y} := \omega_X \otimes f^*\omega_Y^{-1},$$

where ω_X and ω_Y are the canonical sheaves of X and Y.

As before, we apply the Koszul filtration to the sequence (1.22) and to the complex $\bigwedge^\bullet \Omega^1_X(\log \Delta)$, hence we get the exact sequence:

$$0 \to \Omega^{p-1}_{X/Y}(\log \Delta) \otimes f^*\Omega^1_Y(\log S) \to \Omega^p_X(\log \Delta) \to \Omega^p_{X/Y}(\log \Delta) \to 0. \quad (1.23)$$

Remark 1.24. From now on, we will use the exact sequence (1.23) in order to define the sheaf $\Omega^p_{X/Y}(\log \Delta)$.

Let us define semistable families of varieties and give some basic properties of sheaves of relative logarithmic forms of a semistable family. Some other important properties about semistable families will be discussed in Section 3.1.

Definition 1.25. A projective family of varieties $f : X \to Y$ over a curve Y is a semistable family if all fibers are reduced and singular fibers of the family are normal crossing divisors. In the case when $\dim X = 2$, i.e. when $f : X \to Y$ is a semistable family of curves, one requires the additional condition that there is no (-1)-curves in the singular fibers.

Lemma 1.9. *Let $f : X \to Y$ be a semistable family of n-varieties over a curve with discriminant locus $S \subset Y$. Let $\Delta = f^{-1}S$. One has:*

$$\Omega^n_{X/Y}(\log \Delta) = \omega_{X/Y}.$$

Proof. We have to note that for $p = n + 1$, since the relative dimension of the family is n, then $\Omega^{n+1}_{X/Y}(\log \Delta) = 0$ and we get from the previous exact sequence:

$$\Omega^{n+1}_X(\log \Delta) \simeq \Omega^n_{X/Y}(\log \Delta) \otimes f^*\Omega^1_Y(\log S),$$

$$\Omega^n_{X/Y}(\log \Delta) = \Omega^{n+1}_X(\log \Delta) \otimes f^*\Omega^1_Y(\log S)^{-1}.$$

By property 3) from the beginning of the section, one has:

$$\Omega^{n+1}_X(\log \Delta) = \omega_X \otimes \mathcal{O}_X(\Delta),$$

$$\Omega^1_Y(\log S) = \omega_Y \otimes \mathcal{O}_Y(S),$$

and we get: $\Omega^n_{X/Y}(\log \Delta) = \omega_X \otimes f^*\omega_Y^{-1} \otimes \mathcal{O}_X(\Delta) \otimes \mathcal{O}_X(-\Delta) = \omega_{X/Y}$. □

Remark 1.26. It is important to note that for a divisor $T \subset Y$ which contains the discriminant locus S of the semistable family $f : X \to Y$ we can also define the sheaf $\Omega^p_{X/Y}(\log \Delta')$, where $\Delta' = f^*T$, by the sequence:

$$0 \to \Omega^{p-1}_{X/Y}(\log \Delta') \otimes f^*\Omega^1_Y(\log T) \to \Omega^p_X(\log \Delta') \to \Omega^p_{X/Y}(\log \Delta') \to 0.$$

Then arguing as above, we get: $\Omega^n_{X/Y}(\log \Delta') = \omega_{X/Y}$.

Definition 1.27. Let $f : X \to Y$ be a map between two topological spaces. We define the k-th derived functor from the category of sheaves on X to the category of sheaves on Y to be the functor:

$$R^k f_* : Sh(X) \to Sh(Y)$$

$$\mathcal{F} \mapsto R^k f_* \mathcal{F},$$

where $R^k f_* \mathcal{F}$ is the sheaf associated to the presheaf

$$U \to H^k(f^{-1}(U), \mathcal{F}_{|f^{-1}(U)}).$$

The following theorems are Lemma 1.2, Lemma 1.3 from [21] and Lemma 3.21 from [20]. They describe the direct images of sheaves of logarithmic differential forms on finite coverings.

Theorem 1.10. *Let X be a complex projective manifold and $\pi : W_1 \to X$ be a finite covering such that W_1 is normal, D a divisor on X with normal crossings which contains the branch divisor of the covering, $\mu : W \to W_1$ a desingularization map, $\tau : W \to X$ the corresponding composition map and assume that $\Delta = \tau^*D$ is a normal crossing divisor. Then :*

1. $\tau_*\Omega^p_W(\log \Delta) = \Omega^p_X(\log D) \otimes \tau_*\mathcal{O}_W$;
2. $R^q\tau_*\Omega^p_W(\log \Delta) = 0$, $R^q\mu_*\Omega^p_W(\log \Delta) = 0$, $R^q\tau_*(\mathcal{O}_W) = 0$, *for* $q > 0$.

Theorem 1.11. *(Hurwitz's general formula) Let X be a complex projective manifold and $\pi : W_1 \to X$ be a finite covering, D the branch divisor of the covering with normal crossings on X, $\mu : W \to W_1$ a desingularization map, $\tau : W \to X$ the corresponding composition map and assume that $\Delta = \tau^*D$ is a normal crossing divisor. Then, one has an inclusion:*

$$\tau^*\Omega^p_X(\log D) \subset \Omega^p_W(\log \Delta),$$

which is an isomorphism over the open set U of W, where the map $\tau : W \to X$ is finite.

1.3 Cohomology of cyclic coverings

Let $f : X \to Y$ be a semistable family of n-folds over a curve Y. We will suppose that the map $f : X \to Y$ is smooth over $Y \backslash S$, where S is a finite set of points on Y.

Let $\mathcal{M}$ be an invertible sheaf on X, let $\nu \geq 2$ be a positive integer and let $D = \sum_{j=1}^{k} \alpha_j D_j$ be a normal crossing divisor such that $\mathcal{M}^{\nu} \cong \mathcal{O}_X(D)$. Let $\pi : W_1 \to X$ be the ν-cyclic covering over X obtained by taking the ν-th root out of D. In particular, W_1 is normal. Let $\mu : W \to W_1$ be a desingularization of the cyclic covering. We consider the induced maps $\tau = \pi \circ \mu$ and $h = f \circ \tau$. Let T be the reduced divisor on Y, whose support is the union of S and $f(\pi(\mathrm{Sing}(W_1)))$. Let $\Delta = f^*T$ and let as before Γ be the reduced divisor whose support consists of all components of $D = \sum_{j=1}^{k} \alpha_j D_j$, whose coefficients are not divisible by ν. We recall that the divisor Γ is the support of the branch locus of the cyclic covering W_1, due to Lemma 1.5. We will assume that $\Delta + \Gamma$ is a normal crossing divisor, that Δ and Γ have no common components and that the desingularization μ is chosen such that $\tau^*(\Delta + \Gamma)$ is a normal crossing divisor.

Lemma 1.12. *In the set up from the beginning of the section, the cyclic covering W_1 has only rational singularities and $\mu_*\mathcal{O}_W = \mathcal{O}_{W_1}$.*

Proof. This is a consequence of the fact that the branch divisor D of the cyclic covering W_1 is a normal crossing divisor, therefore by the Lemma 1.7 one has that W_1 has only rational singularities and $\mu_*\mathcal{O}_W = \mathcal{O}_{W_1}$. □

Theorem 1.13. *In the set up from the beginning of the section, we have:*

$$\tau_*\Omega^p_W(\log \tau^*(\Delta + \Gamma)) = \bigoplus_{i=0}^{\nu-1} \Omega^p_X(\log(\Delta + \Gamma)) \otimes \mathcal{M}^{(-i)},$$

$$R^q\tau_*(\Omega^p_W(\log \tau^*(\Delta + \Gamma))) = 0$$

for all $p \geq 0$ and $q > 0$. Also, we have that: $R^q\mu_*\mathcal{O}_W = 0$ for any $q > 0$.

Proof. The divisor $\tau^*(\Delta+\Gamma)$ has normal crossings and on W_1 we have at worst rational singularities. By Lemma 1.12 we have: $\mu_*\mathcal{O}_W = \mathcal{O}_{W_1}$. By Theorem 1.2 one has $\pi_*\mathcal{O}_{W_1} = \bigoplus_{i=0}^{\nu-1} \mathcal{M}^{(-i)}$. Hence,

$$\tau_*\mathcal{O}_W = \pi_*\mu_*\mathcal{O}_W = \pi_*\mathcal{O}_{W_1} = \bigoplus_{i=0}^{\nu-1} \mathcal{M}^{(-i)},$$

and the theorem follows from Theorem 1.10. □

In this part we will prove several lemmas about the cohomology of cyclic coverings constructed over a semistable family. These lemmas are the relative-case analogues of Lemma 3.16 d) and Lemma 3.22 in [20]. A sketch of the proof in the relative case is given in the paper [78] Lemma 6.2. Using those indications we prove these results in details.

Let us define the sheaf $\Omega^p_{W/Y}(\log \tau^*(\Delta+\Gamma))$, by induction on p :

$$0 \to \Omega^{p-1}_{W/Y}(\log \tau^*(\Delta+\Gamma)) \otimes h^*\Omega^1_Y(\log T) \to \Omega^p_W(\log \tau^*(\Delta+\Gamma)) \\ \to \Omega^p_{W/Y}(\log \tau^*(\Delta+\Gamma)) \to 0. \tag{1.24}$$

Lemma 1.14. *In the previous notation, we have:*

$$R^q\tau_*(\Omega^p_{W/Y}(\log \tau^*(\Delta+\Gamma))) = 0 \textit{ for all } p \geq 0 \textit{ and } q > 0.$$

Proof. By Theorem 1.10 we have

$$R^q\tau_*(\Omega^p_W(\log \tau^*(\Delta+\Gamma))) = 0, \text{ for all } p \geq 0 \text{ and } q > 0.$$

We will proceed by induction on the degree p of forms to prove the analogous formula for the relative case.

For $p = 0$, we have:

$$R^q\tau_*(\Omega^p_{W/Y}(\log \tau^*(\Delta+\Gamma))) = R^q\tau_*(\mathcal{O}_W) = 0, \forall q > 0,$$

by Theorem 1.10.

Now let us fix $p \neq 0$. The induction hypothesis will be that for all $q > 0$

$$R^q\tau_*(\Omega^{p-1}_{W/Y}(\log \tau^*(\Delta+\Gamma))) = 0.$$

The long exact sequence associated to the sequence (1.24) obtained by taking τ_* and the projection formula yield:

$$0 = R^q\tau_*(\Omega^p_W(\log \tau^*(\Delta+\Gamma))) \to R^q\tau_*(\Omega^p_{W/Y}(\log \tau^*(\Delta+\Gamma))) \to \\ R^{q+1}\tau_*\left(\Omega^{p-1}_{W/Y}(\log \tau^*(\Delta+\Gamma)) \otimes h^*\Omega^1_Y(\log T)\right)$$

and

$$\begin{aligned} &R^{q+1}\tau_*\left(\Omega^{p-1}_{W/Y}(\log \tau^*(\Delta+\Gamma)) \otimes h^*\Omega^1_Y(\log T)\right) \\ &= R^{q+1}\tau_*(\Omega^{p-1}_{W/Y}(\log \tau^*(\Delta+\Gamma))) \otimes f^*\Omega^1_Y(\log T) \\ &= 0. \end{aligned}$$

Hence,

$$R^q\tau_*(\Omega^p_{W/Y}(\log \tau^*(\Delta+\Gamma))) = 0.$$

$\square$

Lemma 1.15. *In the previous notation we have the decomposition:*

$$\tau_*\Omega^p_{W/Y}(\log\tau^*(\Delta+\Gamma)) = \bigoplus_{i=0}^{\nu-1}\Omega^p_{X/Y}(\log(\Delta+\Gamma))\otimes\mathcal{M}^{(-i)} \textit{ for all } p\geq 0.$$

Proof. By Theorem 1.13, we have:

$$\tau_*\Omega^p_W(\log\tau^*(\Delta+\Gamma)) = \bigoplus_{i=0}^{\nu-1}\Omega^p_X(\log(\Delta+\Gamma))\otimes\mathcal{M}^{(-i)}.$$

The proof will be done by induction on the degree of the sheaf of relative differential forms.

In the case when $p=0$, we have:

$$\tau_*\mathcal{O}_W = (\pi\circ\mu)_*\mathcal{O}_W = \pi_*(\mu_*\mathcal{O}_W).$$

On W_1 we have at worst rational singularities, hence:

$$\mu_*\mathcal{O}_W = \mathcal{O}_{W_1}$$

and

$$\tau_*\mathcal{O}_W = \pi_*\mathcal{O}_{W_1} = \bigoplus_{i=0}^{\nu-1}\mathcal{M}^{(-i)}.$$

Now, let us fix $p\neq 0$. The induction hypothesis will be:

$$\tau_*\Omega^{p-1}_{W/Y}(\log\tau^*(\Delta+\Gamma)) = \Omega^{p-1}_{X/Y}(\log(\Delta+\Gamma))\otimes\bigoplus_{i=0}^{\nu-1}\mathcal{M}^{(-i)}.$$

The long exact sequence associated to the sequence (1.24) obtained by taking τ_* is:

$$\begin{aligned}0\to&\tau_*(\Omega^{p-1}_{W/Y}(\log\tau^*(\Delta+\Gamma)\otimes h^*\Omega^1_Y(\log T))\to\tau_*\Omega^p_W(\log\tau^*(\Delta+\Gamma))\\ &\to\tau_*\Omega^p_{W/Y}(\log\tau^*(\Delta+\Gamma))\to R^1\tau_*(\Omega^{p-1}_{W/Y}(\log\tau^*(\Delta+\Gamma)))\otimes f^*\Omega^1_Y(\log T)\to\ldots\end{aligned}$$

By Lemma 1.14 one has:

$$R^1\tau_*(\Omega^{p-1}_{W/Y}(\log\tau^*(\Delta+\Gamma)))=0.$$

This produces the short exact sequence:

$$\begin{aligned}0\to\tau_*(\Omega^{p-1}_{W/Y}(\log\tau^*(\Delta+\Gamma))\otimes f^*\Omega^1_Y(\log T)&\to\tau_*\Omega^p_W(\log\tau^*(\Delta+\Gamma))\\ &\to\tau_*\Omega^p_{W/Y}(\log\tau^*(\Delta+\Gamma))\to 0.\end{aligned}$$

Using the induction hypothesis, for $p-1$, we get:

$$0 \to \left(\Omega^{p-1}_{X/Y}(\log(\Delta+\Gamma)) \otimes \bigoplus_{i=0}^{\nu-1} \mathcal{M}^{(-i)} \right) \otimes f^*\Omega^1_Y(\log T) \to \Omega^p_X(\log(\Delta+\Gamma)) \otimes \bigoplus_{i=0}^{\nu-1} \mathcal{M}^{(-i)}$$
$$\to \tau_*\Omega^p_{W/Y}(\log \tau^*(\Delta+\Gamma)) \to 0.$$

Therefore,

$$\begin{aligned}\tau_*\Omega^p_{W/Y}(\log \tau^*(\Delta+\Gamma)) &= \frac{\Omega^p_X(\log(\Delta+\Gamma) \otimes \bigoplus_{i=0}^{\nu-1} \mathcal{M}^{(-i)}}{\Omega^{p-1}_{X/Y}(\log(\Delta+\Gamma)) \otimes f^*\Omega^1_Y(\log T) \otimes \bigoplus_{i=0}^{\nu-1} \mathcal{M}^{(-i)}} \\ &= \Omega^p_{X/Y}(\log(\Delta+\Gamma)) \otimes \bigoplus_{i=0}^{\nu-1} \mathcal{M}^{(-i)}\end{aligned}$$

where the last equality comes from the definition of $\Omega^p_{X/Y}(\log(\Delta+\Gamma))$. $\square$

Lemma 1.16. *In the previous notation, we have the decomposition:*

$$\tau_*\Omega^p_{W/Y}(\log \tau^*(\Delta)) = \Omega^p_{X/Y}(\log \Delta) \oplus \bigoplus_{i=1}^{\nu-1} \Omega^p_{X/Y}(\log(\Delta+\Gamma_i)) \otimes \mathcal{M}^{(-i)},$$

where $\Gamma_i = \sum\limits_{\frac{\alpha_j i}{\nu} \notin \mathbb{Z}} D_j$.

Proof. By the previous lemma, we have:

$$\tau_*\Omega^p_{W/Y}(\log \tau^*(\Delta+\Gamma)) = \bigoplus_{i=0}^{\nu-1} \Omega^p_{X/Y}(\log(\Delta+\Gamma)) \otimes \mathcal{M}^{(-i)},$$

where $\Gamma = \Gamma_1 = \sum\limits_{\frac{\alpha_j}{\nu} \notin \mathbb{Z}} D_j$ is the support of the branch locus of the cyclic covering. Note that $\Gamma_i \subseteq \Gamma$ for all i.

We will argue locally, outside of the singularities of D. Hence, we can assume that on some small open set $U \subset X \backslash \mathrm{Sing} D$, the divisor $D = \alpha_1 D_1$ where $f_1 = 0$ is a local equation of D_1.

Let $l_i = \phi \cdot m_i$ be a section of the sheaf $\Omega^p_{X/Y}(\log(\Delta+\Gamma)) \otimes \mathcal{M}^{(-i)}$ such that:

$$\phi \in H^0(U, \Omega^p_{X/Y}(\log(\Delta+\Gamma))),$$

and

$$m_i \in H^0(U, \mathcal{M}^{(-i)})$$

are local frames of $\mathcal{M}^{(-i)}$ on U. In the course of the proof of Theorem 1.2 we saw that the space of sections of $\bigoplus_{i=0}^{\nu-1} \mathcal{M}^{(-i)}$ over U is isomorphic to the

B-algebra $\bigoplus_{i=0}^{\nu-1} Bt^i f_1^{-[\frac{i\alpha_1}{\nu}]}$, where $B = H^0(U, \mathcal{O}_U)$ and $t^\nu - uf_1^{\alpha_1} = 0$ is a local equation of the cyclic covering with $u \in B^*$. Then, we can take

$$m_i = t^i f_1^{-[\frac{i\alpha_1}{\nu}]}.$$

One has:

$$m_i^\nu = u^i f_1^{i\alpha_1 - \nu[\frac{i\alpha_1}{\nu}]}.$$

We consider two cases:

1. For i such that $\nu \nmid i\alpha_1$, we have $\Gamma_i = D_1$, so:

$$i\alpha_1 - \nu[\frac{i\alpha_1}{\nu}] \neq 0, m_{i\,|D}^\nu = u^i f_1^{i\alpha_1 - \nu[\frac{i\alpha_1}{\nu}]}{}_{|D} = 0,$$

   since $f_{1|D} = 0$. This implies $m_{i\,|\Gamma_i}^\nu = 0$ and then $m_{i|\Gamma_i} = 0$. So, $l_i = \phi m_i$ has no poles over Γ_i since they will cancel with zeros of m_i on Γ_i.

2. For every i such that $\nu \mid i\alpha_1$, we have $m_i^\nu = u^i$, so m_i is an invertible function. In order that l_i has no pole along Γ_i, ϕ has to belong to $\Omega^p_{X/Y}(\log(\Delta))$.

Since, $\tau_*\Omega^p_{W/Y}(\log \tau^*(\Delta)) \subset \tau_*\Omega^p_{W/Y}(\log \tau^*(\Delta + \Gamma))$ is the direct image of the sheaf of forms without poles over $\tau^*(\Gamma)$, then from the previous analysis of the sheaf

$$\bigoplus_{i=0}^{\nu-1} \Omega^p_{X/Y}(\log(\Delta + \Gamma)) \otimes \mathcal{M}^{(-i)},$$

we conclude that forms which have no poles on Γ are those which lie in

$$\Omega^p_{X/Y}(\log \Delta) \oplus \bigoplus_{i=1}^{\nu-1} \Omega^p_{X/Y}(\log(\Delta + \Gamma_i)) \otimes \mathcal{M}^{(-i)}.$$

□

Chapter 2

Local systems, variations of Hodge structures and Higgs bundles

In this chapter we will give a general picture of the equivalence between local systems, complex variations of Hodge structures and Higgs bundles on a Kähler manifold. We will not explain in details proofs of the results, but rather explain the connection between these notions which will be one of the basic tools used in the next chapters. The main part of the theory stated here can be found in [3], [10], [14], [67] and [81]. In the last section of this chapter we recall some basic facts about Teichmüller space and Teichmüller curves. The aim of this section will be to give the proof of one theorem of Möller which states that if the Higgs field of a subvariation of the geometric variation of a semistable family of curves is an isomorphism then the base curve is a Teichmüller curve in the moduli space of curves of genus g, where g is the genus of a smooth fiber of the family.

2.1 Hodge structures

Smooth projective varieties are Kähler since the Fubini-Study metric on projective space is Kähler and then the pull-back of a Kähler metric to a closed complex projective submanifold is Kähler too. If a manifold is embedded in projective space by means of the sections of a very ample line bundle $\mathcal{L}$, then its Kähler class associated to the pullback of the Fubini-Study metric is the first Chern class $c_1(\mathcal{L})$ or the cycle class of hyperplane sections. It is important to note that not all Kähler manifolds are projective, for example their Kähler class is not an integral class in general. However, the existence of a Kähler metric has a lot of consequences for the cohomology of a complex manifold. We will recall here some important results for Kähler manifolds. Let us first recall the definition of Hodge structures, since these structures lie on the cohomologies of the Kähler manifolds.

Definition 2.1. (Hodge structures [81]) An integral Hodge structure of weight k is given by a free abelian group $V_{\mathbb{Z}}$ of finite type, together with a decomposition:

$$V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q},$$

where $V^{p,q}$ are $\mathbb{C}$-vector spaces, satisfying $V^{p,q} = \overline{V^{q,p}}$. Given such a decomposition, we define the associated Hodge filtration $F^{\bullet}V_{\mathbb{C}}$ by

$$F^p V_{\mathbb{C}} = \bigoplus_{r \geq p} V^{r,k-r}.$$

It is a decreasing filtration on $V_{\mathbb{C}}$ which satisfies

$$F^p V_{\mathbb{C}} \oplus \overline{F^{q+1} V_{\mathbb{C}}} = V_{\mathbb{C}}.$$

The Hodge filtration determines the Hodge decomposition by

$$V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}, p+q=k.$$

Moreover, if there is a non-degenerate bilinear form Q on $V_{\mathbb{C}}$ which is symmetric for k even and alternating otherwise and which is integral on $V_{\mathbb{Z}}$, i.e. $Q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$ and which satisfies Hodge-Riemann conditions:

1. $F^p V_{\mathbb{C}}$ is Q-orthogonal to $F^{q+1} V_{\mathbb{C}}$;
2. $Q(\sqrt{-1}^{p-q} v, v) > 0$ for any nonzero vector $v \in F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}},\ p+q=k$;

then the bilinear form Q is called a polarization of the Hodge structure. Let us sum this up by the next definition:

Definition 2.2. ([81],§7.7) An integral polarized Hodge structure of weight k is given by a Hodge structure $(V_{\mathbb{Z}}, F^p V_{\mathbb{C}})$ of weight k, together with a non-degenerate bilinear form Q on $V_{\mathbb{Z}}$, which is symmetric if k is even, alternating otherwise and satisfies Hodge-Riemann bilinear relations.

The well known examples are the Hodge structures on cohomology groups of Kähler manifolds. Let us recall the theorem:

Theorem 2.1. *(Hodge decomposition [81]) Suppose that X is a compact Kähler manifold of dimension n. Then $\mathbb{C}^*$ acts naturally on the cohomology of X, so that*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q},$$

for any positive integer $k \leq n$, where the action of $z \in \mathbb{C}^*$ on $H^{p,q}$ is given by multiplication by $z^{-p}\overline{z}^{-q}$. Moreover, one has: $H^{p,q} = \overline{H^{q,p}}$. There is a natural identification:

$$H^{p,q} = H^q(X, \Omega_X^p).$$

If we define the decreasing filtration:

$$F^pH^n(X,\mathbb{C}) = \bigoplus_{r\geq p} H^{r,n-r},$$

then we have :

$$H^{p,q} = F^pH^n(X,\mathbb{C}) \cap \overline{F^qH^n(X,\mathbb{C})}.$$

Let X be a Kähler manifold of dimension n with Kähler form ω. The exterior product with ω defines an operator

$$L : \mathcal{A}^k(X) \to \mathcal{A}^{k+2}(X)$$

called the Lefschetz operator, where $\mathcal{A}^k(X)$ is the space of k-differential forms on X. The cup product with the class $[\omega]$ defines the operator

$$L : H^k(X,\mathbb{R}) \to H^{k+2}(X,\mathbb{R}),$$

on the de Rham cohomology groups. Then, we have the operator

$$L^{n-k} : H^k(X,\mathbb{R}) \to H^{2n-k}(X,\mathbb{R}).$$

This operator gives an intersection form on $H^k(X,\mathbb{R})$, for $k \leq n$ defined as :

$$Q(\alpha,\beta) = \langle L^{n-k}\alpha,\beta\rangle = \int_X \omega^{n-k} \wedge \alpha \wedge \beta,$$

where α and β are representative forms of cohomology classes $[\alpha], [\beta]$ from $H^k(X,\mathbb{R})$ and $\langle .,. \rangle$ stands for the pairing $H^k(X,\mathbb{R}) \otimes H^{2n-k}(X,\mathbb{R}) \to \mathbb{R}$. For k odd Q is alternating, otherwise it is symmetric. We have the induced Hermitian form on $H^k(X,\mathbb{C})$ defined as $H(\alpha,\beta) = i^kQ(\alpha,\overline{\beta})$. It satisfies relations:
R1) The Hodge decomposition is orthogonal for H;
R2) $i^{p-q-k}(-1)^{\frac{k(k-1)}{2}}H(\alpha,\alpha) > 0$ for a non-zero form α of type (p,q) from $H^k(X)_{\text{prim}} = \text{Ker}L^{n-k+1}$.

If $[\omega]$ is an integral class, X is a projective variety, and then Q takes integral values on integral classes.

If we restrict the Hermitian form H on $H^k(X)_{\text{prim}}$, then the relations R1 and R2 are called the Hodge-Riemann bilinear relations.

2.2 Local systems and complex variations of Hodge structures

In this section we will define local systems, then we will explain the relation between local systems on a manifold and finite dimensional representations of the fundamental group of that manifold. Also, we will explain the equivalence between local systems and bundles with a flat connection. At the end we will define variations of Hodge structures. For the first part the reference is the book [69] §2.5 and the rest can be found in [3].

Definition 2.3. Let Y be a locally connected topological space. The constant sheaf on Y with stalk F, is the sheaf whose space of sections over any open connected subset $U \subset X$ is isomorphic to F.

Definition 2.4. A sheaf $\mathcal{F}$ on a locally connected topological space Y is a locally constant sheaf if for every point $x \in X$ there is a neighborhood U of x such that the restriction of $\mathcal{F}$ to U is isomorphic to a constant sheaf.

Definition 2.5. Let Y be a topological space. A space over Y is a topological space X together with a continuous map $p : X \to Y$. We say that X is a cover of Y if for each point y of Y there is an open neighborhood U of y for which $p^{-1}(U)$ decomposes as a disjoint union of open sets V_i of X such that the restriction of p to each V_i induces a homeomorphism of V_i with U. The cover $p : X \to Y$ is trivial if $p : X \to Y$ is isomorphic as a space over Y to $p_1 : Y \times I \to Y$, where p_1 is the first projection and I is a discrete topological space.

Now, let us explain the relation between covers and locally constant sheaves. Let Y be a topological locally connected space. Let $p : X \to Y$ be a space over Y. Let $U \subset Y$ be an open set. We define a section of p over U as a continuous map $s : U \to X$ such that $p \circ s = id_U$.

Also, we can define the presheaf $\mathcal{F}_X$ on Y by setting $\mathcal{F}_X(U)$ to be the set of sections of p over U. By Proposition 2.5.8 from [69], the presheaf $\mathcal{F}_X$ is a sheaf. Moreover, if p is a cover map then $\mathcal{F}_X$ is locally constant, if p is a trivial cover then $\mathcal{F}_X$ is a constant sheaf. The rule $(X \to Y) \mapsto \mathcal{F}_X$ defines an equivalence between the category of covers on Y and the category of locally constant sheaves on Y except the constant sheaf that to every open set of Y associates the empty set.

The functor in the reverse direction is $\mathcal{F} \to Y_{\mathcal{F}}$, where $Y_{\mathcal{F}}$ is the disjoint union of stalks $\mathcal{F}_y, y \in Y$, called the *étale space* of the sheaf $\mathcal{F}$. Recall that the stalk $\mathcal{F}_y$ is the disjoint union of the sets $\mathcal{F}(U)$ over all open neighborhoods U of y, modulo the equivalence relation: $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ are equivalent if there exists an open neighborhood $W \subseteq U \cap V$ of y with $s_{|W} = t_{|W}$. The map $p_{\mathcal{F}} : Y_{\mathcal{F}} \to Y$ is induced by constant maps $\mathcal{F}_y \to \{y\}$.

Then, for a locally constant sheaf $\mathcal{F}$ and $y \in Y$ there is an open connected set U (Y is a locally connected space) such that $\mathcal{F}_{|U}$ is isomorphic to the constant sheaf defined by some set F, with discrete topology. Then we have $\mathcal{F}_y = F$ for all $y \in U$, so we get $p_{\mathcal{F}}^{-1}(U) = U \times F$, i.e. $Y_{\mathcal{F}}$ is a cover.

Let us recall the definition of the inverse image of a sheaf. Let $f : X \to Y$ be a continuous map between topological spaces. Let $\mathcal{F}$ be a sheaf on Y. Let $Y_{\mathcal{F}}$ be the étale space of the sheaf $\mathcal{F}$ with the projection map $\pi : Y_{\mathcal{F}} \to Y$. We have the fiber product:

$$Z = Y_{\mathcal{F}} \times_Y X = \{(v, x) \in Y_{\mathcal{F}} \times X | \pi(v) = f(x)\},$$

with the topology induced by the product topology on $Y_{\mathcal{F}} \times X$, and the induced map $p : Z \to X$, which is a local homeomorphism. The sheaf corresponding to the space $Z \to X$ over X is defined by the rule $(Z \to X) \mapsto \mathcal{F}_Z$ (see above). Finally, the inverse image of the sheaf $\mathcal{F}$ is defined as:

$$f^{-1}\mathcal{F} := \mathcal{F}_Z.$$

Lemma 2.2. *Let $f : X \to Y$ be a continuous map between locally connected topological spaces. If $\mathcal{F}$ is a locally constant sheaf on Y then $f^{-1}\mathcal{F}$ is a locally constant sheaf on X.*

Proof. Let $x \in X$, we want to prove that there is a neighborhood U of x such that $(f^{-1}\mathcal{F})_{|U}$ is a constant sheaf.

For $f(x) \in Y$ there is a neighborhood $V_{f(x)}$ such that $\mathcal{F}_{|V_{f(x)}}$ is a constant sheaf defined by some set F. By the equivalence between the category of covers of some locally connected topological space and the category of locally constant sheaves on that space, trivial covers correspond to constant sheaves. Hence, the constant sheaf $\mathcal{F}_{|V_{f(x)}}$ corresponds to the trivial cover $V_{f(x)} \times F \to V_{f(x)}$. The pullback of $\mathcal{F}_{|V_{f(x)}}$ to X corresponds to the trivial cover $f^{-1}(V_{f(x)}) \times F \to f^{-1}(V_{f(x)})$. Hence, $f^{-1}\mathcal{F}_{|f^{-1}(V_{f(x)})}$ is a constant sheaf, i.e. $f^{-1}\mathcal{F}$ is a locally constant sheaf.

□

Definition 2.6. A complex local system on a topological space Y is a locally constant sheaf of finite dimensional complex vector spaces. If Y is connected all stalks must have the same dimension which is called the dimension of the local system.

Lemma 2.3. *A local system $\mathcal{F}$ on the interval $[0,1]$ is constant.*

Proof. Since $[0,1]$ is simply connected, it has only trivial covers. Imposing on $\mathbb{C}$ the discrete topology, one gets that $\mathcal{F}$ is a constant sheaf by the previous equivalence. □

Proposition 2.4. *There is an equivalence between the category of finite dimensional complex representations of the fundamental group of a connected manifold X and the category of complex local systems on X.*

Proof. Let us suppose that $\mathcal{F}$ is a local system on a connected manifold X with stalks $\mathbb{C}^n$. Let γ be a path on X such that $\gamma : [0,1] \to X, \gamma(0) = x_0$ and $\gamma(1) = x_1$. The pull-back $\gamma^{-1}\mathcal{F}$ will be a locally constant sheaf too. On the other hand a locally constant sheaf on $[0,1]$ is actually constant, by the previous lemma. So, fibers at points 0 and 1 are identified. This yields a $\mathbb{C}$-vector space isomorphism:

$$\phi^{\gamma} : \mathcal{F}_{x_0} \to \mathcal{F}_{x_1},$$

which depends only on the homotopy class of the path γ. Taking closed loops based at x_0, we get a map $\rho : \pi_1(X, x_0) \to \mathrm{GL}(\mathcal{F}_{x_0}) \cong \mathrm{GL}(\mathbb{C}, n)$. This is a group homomorphism and it defines a representation of the fundamental group of X to the space $\mathrm{GL}(\mathcal{F}_{x_0})$.

Conversely, let us suppose that $\rho : \pi_1(X, x_0) \to \mathrm{GL}(\mathbb{C}, n)$ is a finite dimensional representation of the fundamental group and $u : \tilde{X} \to X$ the universal covering map. We can define a vector bundle $V \to X$ such that $V = \tilde{X} \times \mathbb{C}^n / \sim$ where the equivalence relation is given by the group action of $\pi_1(X, x_0)$ on $\tilde{X} \times \mathbb{C}^n$ defined as:

$$\gamma(\tilde{x}, z) = (\gamma(\tilde{x}), \rho(\gamma^{-1})z).$$

Now, let $U \subset X$ be an open set on X such that $u^{-1}(U)$ is a disjoint union of open sets $W_j \subset \tilde{X}$, and such that W_j is homeomorphic to U for all j. Let $u_j = u_{|W_j}$, then for any $z \in \mathbb{C}^n$ and any choice of j we have a local section:

$$s(x) = (u_j^{-1}(x), z), x \in U.$$

The section s is a constant local section of the bundle V. We denote by $\mathcal{F}$ the sheaf of constant local sections of V. Clearly, $\mathcal{F}$ is a locally constant sheaf, i.e. a local system. $\square$

Let us now investigate another categorical equivalence, the equivalence between the category of complex local systems and holomorphic bundles with a flat connection, on a connected complex manifold X.

Definition 2.7. A holomorphic connection on a holomorphic vector bundle $\mathcal{F}$ on a complex manifold X is a $\mathbb{C}$-linear operator

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X,$$

such that

$$\nabla(fs) = df \otimes s + f\nabla s,$$

for all holomorphic local functions f and for all local sections s of $\mathcal{F}$. The holomorphic connection ∇ induces maps:

$$\nabla : \mathcal{F} \otimes \Omega^p_X \to \mathcal{F} \otimes \Omega^{p+1}_X, \text{ for all } p \ ,$$

defined as $\nabla(s \otimes \alpha) = \nabla s \wedge \alpha + s \otimes d\alpha$, where α is a local holomorphic p-form on Y. Then, the curvature of the connection ∇ is defined as the $\mathcal{O}_X$-linear operator:

$$R = \nabla \circ \nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^2_X, R \in \Omega^2_X(\mathrm{End}\mathcal{F}).$$

We say that a connection is flat or integrable if $R = 0$.

Remark 2.8. If a connection on a bundle $\mathcal{F}$ is flat then for the corresponding locally free sheaf $\mathcal{F}$ we have a complex :

$$0 \to \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X \to \mathcal{F} \otimes \Omega^2_X \to \ldots,$$

with chain map ∇.

Now, let us suppose that $\mathcal{F}$ is a vector bundle with a flat connection ∇. By [23] Proposition 1.14, we have an open covering of X by small open sets U such that the space of solutions for the differential equation $\nabla s = 0$ over U has dimension equal to the rank of $\mathcal{F}$. Then, the existence of a local solution for the linear differential equation $\nabla s = 0$ implies that the sheaf $\mathcal{L}$ defined as $\mathcal{L} = \mathrm{Ker}(\nabla)$ is a locally constant subsheaf of $\mathcal{F}$ and $\mathcal{F} = \mathcal{L} \otimes \mathcal{O}_X$. We say that $\mathcal{F}$ is the flat bundle associated to the local system $\mathcal{L}$.

On the another hand, if we have a local system $\mathcal{L}$ on X, then it can be associated with a locally free sheaf $\mathcal{F}$ defined as $\mathcal{F} = \mathcal{L} \otimes \mathcal{O}_X$, with a flat connection $\nabla = 1 \otimes d$. Since we have an equivalence between locally free sheaves and vector bundles, we get an equivalence of categories between flat bundles and locally constant sheaves of $\mathbb{C}$-vector spaces.

Definition 2.9. A complex variation of Hodge structures on a complex manifold X consists of:

1. A locally constant sheaf of $\mathbb{C}$-vector spaces $\mathbb{E}$ on X.
2. A bundle $E = \mathbb{E} \otimes \mathcal{O}_X$. This bundle has a decomposition by smooth subbundles $E = \bigoplus_{p+q=k} E^{p,q}$.
3. A finite filtration $\{F^p\}$ on the vector bundle $E = \mathbb{E} \otimes \mathcal{O}_X = \bigoplus_{p+q=k} E^{p,q}$ by holomorphic vector bundles $F^pE = \bigoplus_{r \geq p} E^{r,s}$.
4. A connection, $\nabla : E \to E \otimes \Omega^1_X$ which satisfies Griffiths's transversality property, i.e. $\nabla(F^pE) \subseteq (F^{p-1}E) \otimes \Omega^1_X$, and such that the bundle E is flat with respect to this connection. It is also required that the bundles $\overline{F^q} := \bigoplus_{s \geq q} \overline{E^{r,s}}$ carry an anti-holomorphic structure on which ∇ acts by sending $\overline{F^q}$ to $\overline{F^{q-1}} \otimes \overline{\Omega^1_X}$.

We will use $\mathcal{E}^c = \{\mathbb{E}, E, F^\bullet, \nabla\}$ to denote a complex variation of Hodge structures.

We say that a complex variation of Hodge structures $\mathcal{E}^c = \{\mathbb{E}, E, F^\bullet, \nabla\}$ is polarized if there is a locally constant sheaf of free $\mathbb{Z}$-modules of finite rank $\mathbb{E}_\mathbb{Z}$ such that $\mathbb{E} = \mathbb{E}_\mathbb{Z} \otimes \mathbb{C}$ and a locally constant non-degenerate bilinear form:

$$Q : \mathbb{E}_\mathbb{Z} \otimes \mathbb{E}_\mathbb{Z} \to \mathbb{Z}$$

such that for any $y \in Y$ the induced Q_y is a polarization on the Hodge structure $\mathbb{E}_y$.

Proposition 1.13 from [14] gives a nice property of the local systems which underlie polarized variations of Hodge structures. This is the modified Deligne's semi-simplicity theorem:

Theorem 2.5. *A local system $\mathbb{V}$, underlying a polarized variation of Hodge structures, decomposes uniquely as:*

$$\mathbb{V} = \bigoplus_{i=1}^{r} (\mathbb{V}_i \otimes W_i),$$

where the $\mathbb{V}_i$ are pairwise non-isomorphic irreducible $\mathbb{C}$-local systems and the W_i are non-zero $\mathbb{C}$-vector spaces.

Moreover, the $\mathbb{V}_i$ and the W_i carry polarized variations of Hodge structures, whose tensor product and sum give back the Hodge structure on $\mathbb{V}$. The Hodge structure on the $\mathbb{V}_i$ (and W_i) is unique up to a shift of the bigrading.

Given a polarized complex variation of Hodge structures:

$$\mathcal{E}^c = \{\mathbb{E}_{\mathbb{Z}}, \mathbb{E}, E, F^\bullet, \nabla, Q\}$$

of weight k one can induce a Hermitian metric on the vector bundle E. That metric is induced by the polarization Q of the Hodge structures and it is defined as:

$$\langle u, v\rangle_H = Q(C_y u, \overline{v}) \quad u, v \in E_y$$

where H stands for the metric and the operator C_y is the Weil operator defined as

$$C_y(u) = i^{p-q}u, \text{ for } u \in F^p(E_y) \cap \overline{F^q(E_y)}, p+q = k.$$

Definition 2.10. The metric H is called the Hodge metric on the vector bundle E.

Definition 2.11. A family $f : X \to Y$ is called a family of compact Kähler varieties if all fibers are compact Kähler.

Now, let us construct the local system which is naturally associated to a smooth family of compact Kähler n-varieties $f : X \to Y$. It is obvious that fibers of the family are compact Kähler n-varieties. Since all fibers $X_y = f^{-1}(y),\ y \in Y$ are smooth, they are diffeomorphic and so the cohomology groups $H^k(X_y, \mathbb{C})$ of fibers $X_y,\ y \in Y$ are isomorphic.

Definition 2.12. The sheaf $\mathbb{C}_X$ is the constant sheaf on X with stalk $\mathbb{C}$.

Let $U \subset Y$ be a connected open set. Applying the k-th derived functor of f_* to the constant sheaf $\mathbb{C}_X$ on X we get $R^k f_*\mathbb{C}_X$ to be the sheafification of:

$$U \to H^k(f^{-1}(U), \mathbb{C}_{X|f^{-1}(U)}).$$

The fact that $R^k f_*\mathbb{C}_X$ is a local system on Y is just a consequence of the fact that for $y \in Y$ there is a contractible neigbourhood U such that:

$$H^k(f^{-1}(U), \mathbb{C}) \cong H^k(U \times X_y, \mathbb{C}) \cong H^k(X_y, \mathbb{C}).$$

So we get that $R^k f_*\mathbb{C}_X$ restricted to U is isomorphic to the constant sheaf with stalk $H^k(X_y, \mathbb{C})$.

Following the equivalence between local systems and flat bundles, the corresponding holomorphic bundle with a flat connection corresponding to the local system $R^k f_* \mathbb{C}_X$ is the bundle:

$$E = R^k f_* \mathbb{C}_X \otimes \mathcal{O}_Y.$$

One should note that on fibers, by [81] p.230, 250, we have:

$$E_y = (R^k f_* \mathbb{C}_X \otimes \mathcal{O}_Y)_y \cong H^k(X_y, \mathbb{C}) \otimes (\mathcal{O}_{Y,y}/\mathcal{M}_y) \cong H^k(X_y, \mathbb{C}) \otimes \mathbb{C} = H^k(X_y, \mathbb{C}),$$

where $\mathcal{M}_y$ is the ideal of $\mathcal{O}_{Y,y}$ of the holomorphic functions vanishing at y.

The corresponding flat connection ∇_E is defined to be the unique holomorphic connection with local flat sections the local sections of $R^k f_* \mathbb{C}_X$:

$$\nabla_E(s) = 0 \text{ if and only if } s \in R^k f_* \mathbb{C}_X.$$

Definition 2.13. The connection ∇_E is called the Gauss-Manin connection of the bundle $E = R^k f_* \mathbb{C}_X \otimes \mathcal{O}_Y$, of the family $f : X \to Y$.

Let us recall Proposition 2.28 from [14]. This proposition will be the starting point in the construction of a classical example of a complex variation of Hodge structures which arises from a smooth family of varieties over a curve.

Lemma 2.6. *([14] I Proposition 2.28) Let $f : X \to Y$ be a smooth family. Let V be a complex local system on X and k a positive integer. Then, we have the isomorphism:*

$$R^k f_* V \otimes \mathcal{O}_Y \cong R^k f_*(\Omega^\bullet_{X/Y}(V)),$$

where $\Omega^\bullet_{X/Y}(V) = \Omega^\bullet_{X/Y} \otimes_{\mathbb{C}} V$ and $R^k f_*(\Omega^\bullet_{X/Y}(V))$ is the hypercohomology of the complex $\Omega^\bullet_{X/Y}(V)$.

Proof. From the relative holomorphic Poincaré lemma, the complex $\Omega^\bullet_{X/Y}$ is a resolution of the sheaf $f^{-1}\mathcal{O}_Y$. So tensorising both sides of the resolution:

$$f^{-1}\mathcal{O}_Y \hookrightarrow \Omega^\bullet_{X/Y}$$

with V over $\mathbb{C}$ and looking for cohomology of the sheaf $f^{-1}\mathcal{O}_Y \otimes V$ we get:

$$R^k f_*(f^{-1}\mathcal{O}_Y \otimes V) = R^k f_*(\Omega^\bullet_{X/Y}(V)).$$

The result follows by the projection formula. □

The conditions of the previous lemma are satisfied for the family $f : X \to Y$ and the local system $\mathbb{C}_X$, so we get:

$$E = R^k f_* \mathbb{C}_X \otimes \mathcal{O}_Y \cong R^k f_*(\Omega^\bullet_{X/Y}). \tag{2.1}$$

The right hand side is the relative de Rham cohomology.

The Hodge naive filtration $F^\bullet_{nv}$ on $\Omega^\bullet_{X/Y}$ is defined such that $F^p_{nv}(\Omega^\bullet_{X/Y})$ is the complex which has the same term in degree $i \geq p$ as the complex $\Omega^\bullet_{X/Y}$ and it has zeros as terms in degree less than p. We will use the notation:

$$\Omega^{\bullet \geq p}_{X/Y} := F^p_{nv}(\Omega^\bullet_{X/Y}).$$

The spectral sequence associated to this filtration and to the functor f_* is the Hodge to de Rham spectral sequence, so we have :

$$E_1^{p,q} = R^q f_* \Omega^p_{X/Y} \Rightarrow R^k f_*(\Omega^\bullet_{X/Y}), p+q=k. \tag{2.2}$$

Theorem 2.7. *([16] Theorem 5.5) Let $f : X \to Y$ be a smooth family of compact Kähler n-varieties and let p and q be two positive integers such that $p+q \leq n$. Then, one has:*

1. *The sheaves $R^q f_* \Omega^p_{X/Y}$ are locally free of finite rank compatible with base change.*
2. *The spectral sequence (2.2) degenerates at E_1.*
3. *At every point of Y, the sheaves $R^q f_* \Omega^p_{X/Y}$ and $R^p f_* \Omega^q_{X/Y}$ are of the same rank.*

Let us define a filtration $F^\bullet$ on $R^k f_*(\Omega^\bullet_{X/Y})$ by:

$$F^p(R^k f_*(\Omega^\bullet_{X/Y})) = \operatorname{Im}\left(R^k f_*(\Omega^{\bullet \geq p}_{X/Y}) \to R^k f_*(\Omega^\bullet_{X/Y})\right).$$

In particular, the degeneration of the spectral sequence (2.2) implies that (see [24] p.15):

$$F^p R^k f_*(\Omega^\bullet_{X/Y}) \cong R^k f_*(\Omega^{\bullet \geq p}_{X/Y}). \tag{2.3}$$

Therefore, we get a decreasing filtration $F^\bullet$ on the flat bundle $E = R^k f_*(\Omega^\bullet_{X/Y})$, induced by the naive Hodge filtration. One has that on fibers $E_y = H^k(X_y, \mathbb{C})$ the filtration $F^\bullet$ coincides with the Hodge filtration. Obviously by (2.3) one has:

$$F^p H^k(X_y, \mathbb{C}) = \bigoplus_{s \geq p} H^{s,k-s}(X_y). \tag{2.4}$$

Let us give one other description, the algebraic description, of the Gauss-Manin connection on the bundle $E = R^k f_*(\Omega^\bullet_{X/Y})$. In order to do this, we will again use the Koszul filtration (see Section 1.2) on the complex $\bigwedge^\bullet \Omega^1_{X/Y} = \Omega^\bullet_{X/Y}$ and the exact sequence:

$$0 \to f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0.$$

We get the exact sequence of complexes:

$$0 \to f^*\Omega^1_Y \otimes \Omega^\bullet_{X/Y}[-1] \to F^0/F^2 \to \Omega^\bullet_{X/Y} \to 0. \tag{2.5}$$

The long exact sequence associated to the sequence (2.5) obtained by taking f_* has the boundary map:

$$\delta : R^k f_*(\Omega^\bullet_{X/Y}) \to R^{k+1} f_*(f^*\Omega^1_Y \otimes \Omega^\bullet_{X/Y}[-1]),$$

or by the projection formula

$$\delta : R^k f_*(\Omega^\bullet_{X/Y}) \to R^k f_*(\Omega^\bullet_{X/Y}) \otimes \Omega^1_Y.$$

Theorem 2.8. *([3]§2.C) The boundary map δ coincides with the Gauss-Manin connection ∇_E on the de Rham cohomology bundle $E = R^k f_*(\Omega^\bullet_{X/Y})$.*

Let us show that the Gauss-Manin connection satisfies the transversality property. The i-th level of the Hodge filtration on the exact sequence (2.5) induces the exact sequence:

$$0 \to f^*\Omega^1_Y \otimes \Omega^{\bullet \geq i}_{X/Y}[-1] \to (F^0/F^2)^{\geq i} \to \Omega^{\bullet \geq i}_{X/Y} \to 0. \tag{2.6}$$

and also one has:

$$\nabla_E : R^k f_*(\Omega^{\bullet \geq i}_{X/Y}) \to R^{k+1} f_*(\Omega^{\bullet \geq i}_{X/Y}[-1] \otimes f^*\Omega^1_Y) \cong R^k f_*(\Omega^{\bullet \geq i}_{X/Y}) \otimes \Omega^1_Y.$$

Hence, we get the commutative diagram:

$$\begin{tikzcd}
R^k f_*(\Omega^\bullet_{X/Y}) \arrow[r, "\nabla_E"] & R^k f_*(\Omega^\bullet_{X/Y}[-1]) \otimes \Omega^1_Y \\
R^k f_*(\Omega^{\bullet \geq i}_{X/Y}) \arrow[r, "\nabla_E"] \arrow[u] & R^k f_*(\Omega^{\bullet \geq i}_{X/Y}[-1]) \otimes \Omega^1_Y \arrow[u]
\end{tikzcd}$$

The vertical maps in the diagram have for images $F^i(E)$ and $F^{i-1}(E) \otimes \Omega^1_Y$, due to (2.3). Hence, one gets that:

$$\nabla_E(F^i(E)) \subseteq F^{i-1}(E) \otimes \Omega^1_Y,$$

i.e. ∇_E satisfies the Griffiths's transversality property. As a flat connection ∇_E has the decomposition $\nabla^{1,0}_E + \nabla^{0,1}_E$ where $\nabla^{0,1}_E$ defines the holomorphic structure on the bundle E. The fact that ∇_E satisfies the Griffiths's transversality property implies that $F^p R^k f_*(\Omega^\bullet_{X/Y})$ are holomorphic vector subbundles of $E = R^k f_*(\Omega^\bullet_{X/Y})$ (see [30] p.233), or in other words they are preserved by $\nabla^{0,1}_E$.

One has:

$$E_\infty^{p,q} := \mathrm{Gr}^p(E) = \frac{F^p(E)}{F^{p+1}(E)},\ p+q=k.$$

The degeneration of the spectral sequence (2.2) at E_1 implies that:

$$E_1 = E_\infty, \tag{2.7}$$

and we get:

$$\mathrm{Gr}^p(E) = R^q f_* \Omega^p_{X/Y}, \quad p+q=k. \tag{2.8}$$

The Hodge bundles of the family $f : X \to Y$ are defined as:

$$\mathcal{H}^{p,q} := \mathrm{Gr}^p(E) = R^q f_* \Omega^p_{X/Y}, p+q=k.$$

These bundles are holomorphic vector bundles.

Since the filtration $F^\bullet$ coincides on fibers with the Hodge filtration and $E_y = H^k(X_y, \mathbb{C})$ (see (2.4)), one gets:

$$(\mathcal{H}^{p,q})_y = \frac{F^p(E_y)}{F^{p+1}(E_y)} = \frac{\bigoplus_{r\geq p} H^{r,k-r}(X_y)}{\bigoplus_{r\geq p+1} H^{r,k-r}(X_y)} = H^{p,q}(X_y) = H^q(X_y, \Omega^p_{X_y}).$$

In general there is no holomorphic isomorphism between E and $\bigoplus_{p+q=k} \mathcal{H}^{p,q}$, but there is a $\mathcal{C}^\infty$-isomorphism between these bundles. Obviously, this isomorphism induces the Hodge decomposition on fibers.

Bringing together all these data, we get a complex variation of Hodge structures of weight k associated to a smooth family of Kähler varieties $f : X \to Y$, usually called a geometric variation. Moreover, if we suppose that $f : X \to Y$ is a projective family then we get a polarized complex variation of Hodge structures.

Definition 2.14. Let $f : X \to Y$ be a smooth family of Kähler n-varieties. The associated geometric variation of Hodge structures of weight $k \leq n$ on the curve Y, consists of:

1. The local system $R^k f_* \mathbb{C}_X$;
2. The vector bundle $E = R^k f_* \mathbb{C}_X \otimes \mathcal{O}_Y$. This bundle has a decomposition by Hodge bundles $\mathcal{H}^{p,q}$, i.e. one has $\mathcal{C}^\infty$-isomorphism between E and $\bigoplus_{p+q=k} \mathcal{H}^{p,q}$.
3. A decreasing holomorphic filtration $F^\bullet$ on E by holomorphic subbundles $F^pE = \bigoplus_{r\geq p} \mathcal{H}^{r,s}$. On the fibers X_y of the family, we have $E_y = H^k(X_y, \mathbb{C})$ and $F^\bullet E_y$ is the Hodge filtration on $H^k(X_y, \mathbb{C})$. One has $F^p \cap \overline{F^{q+1}} = 0, p+q=k$;
4. An integrable connection, the Gauss-Manin connection $\nabla_E : E \to E \otimes \Omega^1_Y$ which satisfies Griffiths's transversality property:

$$\nabla_E(F^pE) \subseteq (F^{p-1}E) \otimes \Omega^1_Y.$$

One has $\nabla_E(s) = 0$, for all sections of $R^k f_* \mathbb{C}_X$.

5. Moreover, if we suppose that X is projective then the Chern class of the hyperplane bundle restricted to X_y induces integral classes $[\omega_y] \in H^{1,1}(X_y) \cap H^2(X_y, \mathbb{Z})$ which polarize the fibers X_y. They fit together giving a section ω of $R^2 f_* \mathbb{C}$ over Y which induces a bilinear form Q satisfying Hodge-Riemann bilinear relations on the primitive parts of cohomology groups (see Section 2.1).

2.3 Higgs bundles

In this section we will study Higgs bundles on compact Kähler manifolds and explain their relation with flat bundles. The main part in this section is based on Simpson's paper [68] "Higgs bundles and local systems".

Definition 2.15. Let X be a Kähler manifold. A Higgs bundle on X is a pair consisting of a holomorphic bundle $\mathcal{H}$ on X and a holomorphic bundle morphism $\Theta : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_X$, called the Higgs field, which satisfies the condition of integrability $\Theta \wedge \Theta = 0$.

A Higgs bundle $\mathcal{H}$ can be seen as a C^∞-bundle with some operator $\bar{\partial} + \Theta$ where $\bar{\partial}$ defines a holomorphic structure on $\mathcal{H}$ by sending the sections of $\mathcal{H}$ to $(0,1)$-forms with coefficients in $\mathcal{H}$ and Θ is the Higgs field. Using this interpretation of Higgs bundles we will see how by assuming the existence of a harmonic metric on a flat bundle we produce a Higgs bundle.

Let X be a Kähler manifold. Let $(\mathcal{H}, D)$ be a smooth vector bundle on X with a flat connection D, i.e. $(D)^2 = 0$. Let K be a Hermitian metric on the bundle $\mathcal{H}$. We have the decomposition of the connection

$$D = D^{1,0} + D^{0,1},$$

into operators of type $(1,0)$ and $(0,1)$. Let δ' and δ'' be the unique operators of type $(1,0)$ and $(0,1)$, such that connections $\delta' + D^{1,0}$ and $D^{0,1} + \delta''$ preserve the metric K, i.e. for all sections s, t we have:

$$(D^{0,1}s, t)_K + (s, \delta' t)_K = D^{0,1}(s,t)_K,$$

$$(\delta'' s, t)_K + (s, D^{1,0}t)_K = D^{1,0}(s,t)_K.$$

Let $\partial = \dfrac{D^{1,0} + \delta'}{2}, \bar{\partial} = \dfrac{D^{0,1} + \delta''}{2}, \Theta = \dfrac{D^{1,0} - \delta'}{2}$ and $\bar{\Theta} = \dfrac{D^{0,1} - \delta''}{2}$.

Let $D^{0,1}_K = \bar{\partial} + \Theta$. The operator $D^{0,1}_K$ satisfies Leibniz rule, i.e.:

$$D^{0,1}_K(fs) = \bar{\partial}(f) \otimes s + f D^{0,1}_K(s),$$

hence it is a connection. Now, in order to obtain a Higgs bundle we want to impose certain conditions which make the operator integrable, i.e. $(D^{0,1}_K)^2 = 0$.

Let $D_K^{1,0} = \partial + \bar{\Theta}$, then $D = D_K^{1,0} + D_K^{0,1}$. Also, let

$$D_K^c = D_K^{0,1} - D_K^{1,0} = \delta'' - \delta',$$

we get:

$$D_K^{0,1} = \frac{D + D_K^c}{2}.$$

We denote $G_K = (D_K^{0,1})^2 = \dfrac{D^2 + DD_K^c + D_K^c D + {D_K^c}^2}{4}$. Using $D^2 = 0$, we get:

$$(D^{1,0})^2 = 0,\ (D^{0,1})^2 = 0,\ D^{1,0}D^{0,1} + D^{0,1}D^{1,0} = 0,$$

and so

$$(\delta')^2 = (\delta'')^2 = \delta'\delta'' + \delta''\delta' = 0,$$

hence,

$$(D_K^c)^2 = 0.$$

In the end we get:

$$G_K = \frac{DD_K^c + D_K^c D}{4}.$$

From now on we suppose that X is a compact Kähler manifold.

Definition 2.16. The metric K is harmonic if $\Lambda G_K = 0$, where

$$\Lambda : A^{p,q}(X) \to A^{p-1,q-1}(X)$$

is the adjoint operator to the operator

$$L : A^{p,q}(X) \to A^{p+1,q+1}(X)$$

defined as $L(\eta) = \eta \wedge \omega$, where ω is the Kähler form on X.

If we impose the condition that K is a harmonic metric then $G_K = 0$, by [68] p.16. Then for K a harmonic metric, we get $D_K^{0,1}$ integrable:

$$(D_K^{0,1})^2 = 0.$$

Hence,

$$\bar{\partial}^2 = 0,\ \bar{\partial}\Theta = 0,\ \Theta \wedge \Theta = 0, \tag{2.9}$$

or equivalently $\mathcal{H}$ has a holomorphic structure (by Newlander-Nirenberg theorem), then Θ is holomorphic and Θ satisfies the condition of integrability. Hence, a harmonic metric K on $\mathcal{H}$ gives rise to a Higgs bundle $(\mathcal{H}, \Theta)$, whose holomorphic structure is defined by the operator $\bar{\partial}$.

Conversely, let $(\mathcal{H}, \Theta, \bar{\partial})$ be a Higgs bundle with a metric K. Since $\mathcal{H}$ is holomorphic then $\bar{\partial}$ is the operator which sends the holomorphic forms to zero forms. Let $D' = \partial + \bar{\Theta}$ and let $D'' = \bar{\partial} + \Theta$, where ∂ is the unique operator such that $\partial + \bar{\partial}$ preserves the metric K and $\bar{\Theta}$ is defined by:

$$(\Theta(s), t)_K = (s, \bar{\Theta}(t))_K.$$

Then, $D = D' + D'' = \partial + \bar{\partial} + \Theta + \bar{\Theta}$ is a connection on $\mathcal{H}$. If K is chosen such that $D^2 = 0$, then the Higgs bundle $(\mathcal{H}, \Theta, \bar{\partial})$ gives rise to a flat bundle. By [68] one has $D^2 = 0$ if K is a harmonic metric. One of the main results in Simpson's paper [67] §1 is that:

These constructions are inverse: if $(\mathcal{H}, D)$ is a flat bundle and K is a harmonic metric, then the same metric K is a harmonic metric for the resulting Higgs bundle; and the flat connection induced by this metric on the Higgs bundle is equal to the original. Similarly, if $(\mathcal{H}, \Theta)$ is a Higgs bundle and K is a harmonic metric, then the same metric is a harmonic metric on the flat bundle, and the resulting Higgs structure is the same as before.

One can define the degree of a bundle $\mathcal{H}$ on a compact Kähler manifold X of dimension n, with Kähler form ω, as:

$$\deg \mathcal{H} = c_1(\mathcal{H}) \cdot [\omega]^{n-1}.$$

The fact that on a Higgs bundle $(\mathcal{H}, \Theta)$ there exists a harmonic metric K implies that $\mathcal{H}$ has vanishing first and second Chern classes, by [66] p.16. In other words, the degree of the bundle $\mathcal{H}$ is zero. However, that is not the only consequence of K being harmonic. In order to see it, let us first define the semistability of a Higgs bundle:

Definition 2.17. A Higgs bundle $(\mathcal{H}, \Theta)$ on a compact Kähler manifold X with Higgs field

$$\Theta : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_X,$$

is said to be semistable if for any subbundle $\mathcal{F}$ of $\mathcal{H}$ such that $\Theta(\mathcal{F}) \subseteq \mathcal{F} \otimes \Omega^1_Y$, we have

$$\mu(\mathcal{F}) = \frac{\deg \mathcal{F}}{\text{rank } \mathcal{F}} \leq \frac{\deg \mathcal{H}}{\text{rank } \mathcal{H}} = \mu(\mathcal{H}).$$

If the inequality is strict for every proper subbundle $\mathcal{F}$ then $\mathcal{H}$ is called a stable Higgs bundle. The ratio $\mu(\mathcal{F})$ is called the slope of the subbundle $\mathcal{F}$. We say that $(\mathcal{H}, \Theta)$ is a polystable Higgs bundle if it is a direct sum of stable Higgs subbundles of the same slope.

Now, we will give Simpson's theorem which relates the polystability to the existence of harmonic metrics.

Theorem 2.9. *([68] p.19) A Higgs bundle $(\mathcal{H}, \Theta)$ has a harmonic metric if and only if it is polystable and:*

$$c_1(\mathcal{H}) \cdot [\omega]^{n-1} = 0,$$

$$c_2(\mathcal{H}) \cdot [\omega]^{n-2} = 0.$$

Remark 2.18. As a consequence, one can note that there is a natural equivalence between flat bundles with harmonic metric and polystable Higgs bundles of degree 0.

Definition 2.19. We say that $g : (\mathcal{H}, \Theta_H) \to (\mathcal{F}, \Theta_{\mathcal{F}})$ is a morphism of Higgs bundles on X if the following diagram commutes:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\Theta_H"] \arrow[d, "g"] & \mathcal{H} \otimes \Omega^1_X \arrow[d, "g \otimes id"] \\ \mathcal{F} \arrow[r, "\Theta_{\mathcal{F}}"] & \mathcal{F} \otimes \Omega^1_X \end{tikzcd}$$

2.3.1 Examples of Higgs bundles

After having described the equivalence between flat bundles and local systems, we explain now the relation between local systems and Higgs bundles. In general it is impossible to describe the Higgs bundle $(\mathcal{H}, \Theta)$ explicitly in terms of the corresponding local system. However, it is more concrete in the case when a local system corresponds to a polarized complex variation of Hodge structures.

Recall that a polarized complex variation of Hodge structures on a complex manifold Y is a set of data $\mathcal{E}^c = \{\mathbb{E}, E, F^\bullet E, \nabla, Q\}$, where $\mathbb{E}$ is a local system of rank k, E the corresponding flat bundle which has $\mathcal{C}^\infty$-decomposition $\bigoplus_{p+q=k} E^{p,q}$, $F^\bullet$ a finite decreasing filtration on E by holomorphic subbundles $F^pE = \bigoplus_{s\geq p} E^{s,k-s}$, ∇ a flat connection and Q a polarization. Let us define the bundle $\mathcal{H}$ as:

$$\mathcal{H} = \bigoplus_{p+q=k} \mathcal{H}^{p,q},$$

where

$$\mathcal{H}^{p,q} = \mathrm{Gr}^p(E) = F^p(E)/F^{p+1}(E).$$

Note that the bundles E and $\mathcal{H}$ are isomorphic as $\mathcal{C}^\infty$-bundles which enables one to consider ∇ as a connection on $\mathcal{H}$.

The polarization form Q of $\mathcal{E}^c$ induces a Hodge metric H on E in the sense of Definition 2.10. The vector bundles $\mathcal{H}$ and $\{\mathcal{H}^{p,q}\}_{p+q=k}$ are endowed with Hermitian metrics induced by H, which are called the Hodge metrics. Hence, one can consider the second fundamental forms (see Section 4.1 for the definition) of the subbundles $\mathcal{H}^{p,q}$ with respect to the Hodge metric. Let us denote the second fundamental form of $\mathcal{H}^{p,q}$ by $\Theta^{p,q}$. The Griffiths's transversality property implies that

$$\Theta^{p,q} : \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \Omega^1_Y.$$

The sum of the operators $\Theta^{p,q}$ gives a $\mathcal{C}^\infty$-linear operator:

$$\Theta : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_Y.$$

The Griffiths's transversality property induces the decomposition:

$$\nabla = \Theta + \nabla_H + \bar{\Theta},$$

where ∇_H is a connection on $\mathcal{H}$ and $\bar{\Theta}$ is H-conjugate to Θ. By Theorem 13.1.5 from [10] ∇_H is the Chern connection of the metric H on $\mathcal{H}$, i.e. the unique connection of the metric H whose $(0,1)$-part is the operator $\overline{\partial}$. Hence, $\nabla_H = \partial + \bar{\partial}$.

The equation $\nabla^2 = 0$ implies that the operator $\bar{\partial}$ induces a holomorphic structure on $\mathcal{H}$ such that each $\mathcal{H}^{p,q}$ is a holomorphic subbundle of $\mathcal{H}$ and Θ is a holomorphic morphism which satisfies the integrability condition, i.e. $\Theta \wedge \Theta = 0$. Indeed, Θ is the Higgs field of the Higgs bundle $\mathcal{H}$. The Griffiths's transversality property can be now described as:

$$\nabla = \partial + \bar{\partial} + \Theta + \bar{\Theta} : \mathcal{H}^{p,q} \to (\mathcal{H}^{p,q} \otimes \Omega^1_Y) \oplus (\mathcal{H}^{p,q} \otimes \overline{\Omega^1_Y}) \oplus (\mathcal{H}^{p-1,q+1} \otimes \Omega^1_Y) \oplus (\mathcal{H}^{p+1,q-1} \otimes \overline{\Omega^1_Y}). \tag{2.10}$$

Then, the Higgs field Θ can be seen as a part of $\mathrm{Gr}(\nabla)$ or as a direct sum of $\mathcal{O}_Y$-linear morphisms:

$$\Theta_{p,q} : \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \Omega^1_Y,$$

induced by the connection ∇.

Let us give the definition of a harmonic metric on a Higgs bundle over a manifold which is not necessary a Kähler manifold.

Definition 2.20. ([10] §13.1.) A Hermitian metric k on a Higgs bundle $(\mathcal{H}, \Theta)$ is called harmonic if its Chern connection D_k combines with Θ and $\overline{\Theta}$ to give a flat connection $\Theta + D_k + \overline{\Theta}$.

This definition of harmonic metric k has the same implication as the definition of harmonic metric seen in the previous section, i.e. it requires (implies) the integrability of $\Theta + D_k + \overline{\Theta}$. As we saw above, the Hodge metric H on the Higgs bundle $\mathcal{H}$ associated to a polarized complex variation is harmonic. Moreover, if we suppose that Y is a Kähler manifold then the Higgs bundle $\mathcal{H}$ is polystable of degree zero.

Let us give the example of a Higgs bundle arising from the geometric variation of Hodge structures which comes with a smooth family $f : X \to Y$ of projective n-varieties over a curve. Let $k \leq n$ be a positive integer, then the associated geometric variation of Hodge structures of weight k (see Section 2.2) to the family consists of:

- the local system $R^k f_* \mathbb{C}_X$;
- the bundle $E = R^k f_* \mathbb{C}_X \otimes \mathcal{O}_Y \cong R^k f_*(\Omega^\bullet_{X/Y})$;
- the flat connection ∇_E;
- the holomorphic filtration $F^\bullet$ induced by the Hodge naive filtration on the complex $\Omega^\bullet_{X/Y}$;
- the natural polarization Q on fibers of E.

As above, the Higgs bundle associated to this polarized complex variation of Hodge structures is defined as:

$$\mathcal{H} = \bigoplus_{p+q=k} \mathcal{H}^{p,q},$$

where by Section 2.2

$$\mathcal{H}^{p,q} := F^p(E)/F^{p+1}(E) \cong R^q f_* \Omega^p_{X/Y}.$$

By Theorem 2.8, the Gauss-Manin connection

$$\nabla_E : R^k f_*(\Omega^\bullet_{X/Y}) \to R^k f_*(\Omega^\bullet_{X/Y}) \otimes \Omega^1_Y$$

is the edge morphism of the long exact sequence obtained by taking f_* of the sequence:

$$0 \to f^*\Omega^1_Y \otimes \Omega^\bullet_{X/Y}[-1] \to F^0/F^2 \to \Omega^\bullet_{X/Y} \to 0. \tag{2.11}$$

On the p-th level of the exact sequence (2.11) we get the exact sequence:

$$0 \to f^*\Omega^1_Y \otimes \Omega^{p-1}_{X/Y} \to \Omega^p_X \to \Omega^p_{X/Y} \to 0. \tag{2.12}$$

The edge morphisms of the long exact sequence associated to (2.12) obtained by taking f_* are:

$$\Theta_{p,q} : R^q f_* \Omega^p_{X/Y} \to R^{q+1} f_*(\Omega^{p-1}_{X/Y} \otimes f^*\Omega^1_Y) = R^{q+1} f_*(\Omega^{p-1}_{X/Y}) \otimes \Omega^1_Y, \quad p+q=k.$$

These edge morphisms are induced by the Gauss-Manin connection ∇_E on the graded parts of the bundle $R^k f_*(\Omega^\bullet_{X/Y})$. Hence, as we saw in the previous example of Higgs bundle arising from a polarized complex variation of Hodge structures (see 2.10), the sum of $\Theta_{p,q}$'s gives the Higgs field of the Higgs bundle $\mathcal{H}$. Finally, we get the couple:

$$\left(\mathcal{H} = \bigoplus_{p+q=k} R^q f_* \Omega^p_{X/Y}, \Theta = \sum_{p+q=k} \Theta_{p,q}\right),$$

the Higgs bundle and its Higgs field associated to the family $f : X \to Y$.

2.3.2 Logarithmic Higgs bundles. The Deligne extension.

In the previous section we defined the Higgs bundles which arise from a polarized complex variation on a compact manifold and especially on curves and Higgs bundles associated to smooth families of manifolds. The following step will be to consider Higgs bundles on punctured curves and Higgs bundles which are associated to families with singular fibers. These types of Higgs bundles are called logarithmic Higgs bundles since their Higgs fields are singular with at most logarithmic poles at the punctures.

From now on we suppose that the set of punctures S is a finite set of points on a compact smooth curve Y.

Definition 2.21. A logarithmic Higgs bundle $(\mathcal{H}, \Theta)$ on a compact smooth curve Y with respect to S is a locally free sheaf $\mathcal{H}$ on Y, with a Higgs field, that is an $\mathcal{O}_Y$-linear morphism:

$$\Theta : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_Y(\log S),$$

i.e. with at most logarithmic poles at points of the set S.

As we saw in the smooth case, starting with a complex variation of Hodge structures one can construct a Higgs bundle. In the logarithmic framework the construction remains pretty much the same as in the smooth case. But some general properties of these Higgs bundles will depend on the behavior of the monodromy operators around the punctures.

Let $\mathbb{V}$ be a a local system on the curve $Y \backslash S$. The monodromy operators around points of S are defined as:

$$T_s = \rho([\alpha]), s \in S$$

where

$$\rho : \pi_1(Y \backslash S, y_0) \to \mathrm{GL}(\mathbb{V}_{y_0}), y_0 \in Y \backslash S$$

is the corresponding representation and $[\alpha] \in \pi_1(Y \backslash S, y_0)$ is a class of loops around a point $s \in S$.

Definition 2.22. We say that the monodromy around a point $s \in S$ is quasi-unipotent if $(T^l - I)^p = 0$ for some positive integers l, p where I is the identity. If $(T - I)^m = 0$, for some positive integer m, then the monodromy around s is said to be unipotent.

Now, let us give the description of the Higgs bundle which arises from a complex variation of Hodge structures on a punctured curve Y :

Let $\mathcal{V}^c = \{\mathbb{V}, V, F^\bullet(V), \nabla\}$ be a complex variation of Hodge structures of weight k on $Y \backslash S$ with unipotent monodromies around points of S. The flat bundle V is defined as $V = \mathbb{V} \otimes \mathcal{O}_{Y \backslash S}$. Let us state the Deligne's extension theorem:

Theorem 2.10. *([14] II Proposition 5.2) In the unipotent case, the flat bundle $V = \mathbb{V} \otimes \mathcal{O}_{Y \setminus S}$ can be uniquely extended to a bundle E on Y such that the Gauss-Manin connection ∇ has at most logarithmic poles over S. The extension E is called the canonical Deligne extension.*

The extension of the Gauss-Manin connection ∇ which has at most logarithmic singularities over points of S is denoted by ∇_E. The filtration F extends to a holomorphic filtration F_E on E. This gives rise to a logarithmic Higgs bundle $\mathcal{H}$ defined by:

$$\mathcal{H} = \bigoplus_{p+q=k} \mathcal{H}^{p,q} \tag{2.13}$$

where

$$\mathcal{H}^{p,q} = F_E^p(E)/F_E^{p+1}(E),$$

and with Higgs field

$$\Theta : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_Y(\log S),$$

a sum of $\mathcal{O}_Y$-linear morphisms:

$$\Theta_{p,q} : \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \Omega^1_Y(\log S)$$

induced by the extension ∇_E of the Gauss-Manin connection.

Besides the fact that the extension of a flat bundle corresponding to some local system on a punctured curve, with unipotent monodromies around punctures, is canonical, there is also one other advantage to work in the unipotent case. In fact, in the unipotent case the degree of the corresponding Higgs bundle is zero. We next define the polystability of logarithmic Higgs bundles, filtered Higgs bundles and at the end we will state Simpson's correspondence theorem between polystable local systems of degree 0 and polystable Higgs bundles of degree 0. The definition of polystability for the logarithmic Higgs bundles is analogous to the definition of the polystability in the smooth case.

Definition 2.23. A logarithmic Higgs bundle $(\mathcal{H}, \Theta)$ on a smooth compact curve Y with Higgs field

$$\Theta : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_Y(\log S),$$

is said to be semistable if for any subbundle $\mathcal{F}$ of $\mathcal{H}$ such that $\Theta(\mathcal{F}) \subseteq \mathcal{F} \otimes \Omega^1_Y(\log S)$, we have

$$\mu(\mathcal{F}) = \frac{\deg \mathcal{F}}{\text{rank } \mathcal{F}} \leq \frac{\deg \mathcal{H}}{\text{rank } \mathcal{H}} = \mu(\mathcal{H}).$$

When the inequality is strict for any proper subbundle $\mathcal{F}$ of $\mathcal{H}$, then the Higgs bundle $\mathcal{H}$ is said to be stable. The ratio $\mu(\mathcal{F})$ is called the slope of the subbundle $\mathcal{F}$. We say $(\mathcal{H}, \Theta)$ is a polystable logarithmic Higgs bundle if it is a direct sum of stable Higgs subbundles of the same slope.

Definition 2.24. ([66] p.717) Let Y be a smooth compact curve and let S be a finite set of points on Y. Let $\mathcal{E}$ be an algebraic vector bundle on $Y\backslash S$ and let $\mathcal{E}_{\alpha,s}$ be a collection of extensions across the punctures $s \in S$ such that the extensions form a decreasing left continuous filtration, i.e.

$$\mathcal{E}_{\alpha,s} \subset \mathcal{E}_{\beta,s} \text{ for } \alpha \geq \beta,$$

$$\mathcal{E}_{\alpha-\epsilon,s} = \mathcal{E}_{\alpha,s} \text{ for small } \epsilon > 0$$

and if z is a local coordinate vanishing at order one at s, then $\mathcal{E}_{\alpha+1,s} = z\mathcal{E}_{\alpha,s}$. The filtration is described by the α such that $0 \leq \alpha < 1$. The fiber $\mathcal{E}_s$ is a vector space together with a filtration $\mathcal{E}_{\alpha,s}$. Then $(\mathcal{E}, \mathcal{E}_{\alpha,s})_{s\in S}$ is called a filtered vector bundle denoted as $\mathcal{E}_f$. The degree of $\mathcal{E}_f$ is defined by:

$$\deg \mathcal{E}_f = \deg \mathcal{E} + \sum_{s\in S} \sum_{0\leq\alpha<1} \alpha \dim(\mathrm{Gr}_\alpha(\mathcal{E}_s)).$$

Definition 2.25. ([66] p.717) Let Y be a smooth compact curve and let S be a finite set of points on Y. A regular filtered Higgs bundle $\mathcal{H}_f = ((\mathcal{H}, \Theta), (\mathcal{H}_{\alpha,s})_{s\in S})$ consists of:

- a logarithmic Higgs bundle $(\mathcal{H}, \Theta)$ on Y;
- a filtered vector bundle $(\mathcal{H}, \mathcal{H}_{\alpha,s})_{s\in S}$;

such that the condition of regularity is satisfied:

$$\Theta : \mathcal{H}_{\alpha,s} \to \mathcal{H}_{\alpha,s} \otimes \Omega^1_Y(\log s).$$

The degree of $\mathcal{H}_f$ is defined as the degree of the filtered vector bundle $(\mathcal{H}, (\mathcal{H}_{\alpha,s})_{s\in S})$.

Remark 2.26. In the case when the only jump is at $\alpha = 0$ at a point $s \in S$ the filtration of $\mathcal{H}_s$ is said to be trivial. If for every $s \in S$ the filtrations of $\mathcal{H}_s$ are trivial then $\deg \mathcal{H}_f = \deg \mathcal{H}$.

Let us consider a local system $\mathbb{V}$ on a punctured curve $Y\backslash S$. We want to define the stalks of $\mathbb{V}$ at points of punctures. This definition is taken from [67] p. 718 . Let $s \in S$ be a puncture, we fix a ray r_s emanating from the point s. Now, we can consider the stalk of $\mathbb{V}$ over r_s. We define the stalk of $\mathbb{V}$ at the point s as the stalk of $\mathbb{V}$ over r_s. If r_s is extended to a path back to some base point $y \in Y\backslash S$, then $\mathbb{V}_s$ is identified with $\mathbb{V}_y$.

Definition 2.27. A filtered local system $\mathbb{V}$ on the punctured curve $Y\backslash S$, is a local system $\mathbb{V}$ together with decreasing filtrations $\mathbb{V}_{\beta,s}$ of the stalks $\mathbb{V}_s$, indexed by real numbers β. The degree of a filtered local system is defined to be

$$\deg(\mathbb{V}) = \sum_s \sum_\beta \beta \dim(\mathrm{Gr}_\beta(\mathbb{V}_s)).$$

Remark 2.28. When the only jump at a point $s \in S$ is at $\beta = 0$, the filtration of $\mathbb{V}_s$ is said to be trivial.

Let us state the theorem of Simpson about the correspondence between filtered local systems of degree 0 and filtered regular Higgs bundles of degree 0, which can be found in [67] §6:

Theorem 2.11. *(Simpson's correspondence theorem) There exists a natural equivalence between the category of direct sums of stable filtered regular Higgs bundles of degree zero, and of direct sums of stable filtered local systems of degree zero.*

Let us restrict the attention to the case of local systems which come from complex variations of Hodge structures on a punctured curve $Y \backslash S$. We will prove later that these local systems have trivial filtrations of the stalks at points $s \in S$ and hence degree 0. By the previous theorem, the corresponding filtered regular Higgs bundles have degree 0. One could ask the question how we construct the corresponding filtered regular Higgs bundles to local systems underlying complex variations. As it is indicated in [67] p.721, this problem was first solved by Griffiths and Schmid, who defined the corresponding filtered regular Higgs bundle $\mathcal{H}_f$ in the following way:

- $(\mathcal{H}, \Theta)$ is the Higgs bundle defined by (2.13);
- the extensions $(\mathcal{H}_{\alpha,s})$ are compatible with the Hodge decomposition.

Their contributions to this topic were the proofs of the existence of the extensions and that Θ satisfies the condition of regularity. The polystability of the filtered regular Higgs bundle $\mathcal{H}_f$ is just a consequence of the fact that it corresponds to a complex variation of Hodge structures, see [66] §8.

Lemma 2.12. *Let $\mathcal{V}^c = \{\mathbb{V}, V, F^\bullet(V), \nabla\}$ be a complex variation of Hodge structures of weight k on a smooth punctured curve $Y \backslash S$. Then:*

1. *The local system $\mathbb{V}$ has trivial filtration and* $\deg \mathbb{V} = 0$.
2. *If the monodromies around points of S are unipotent, the logarithmic Higgs bundle $(\mathcal{H}, \Theta)$ corresponding to $\mathcal{V}^c$ is polystable of degree zero with trivial filtrations at points of S.*
3. *If the degree of the logarithmic Higgs bundle $(\mathcal{H}, \Theta)$ corresponding to $\mathcal{V}^c$ is zero, then the monodromies around points of S of the local system $\mathbb{V}$ are unipotent.*

Proof. Let $\mathcal{H}_f = ((\mathcal{H}, \Theta), (\mathcal{H}_{\alpha,s}))$ be the filtered regular Higgs bundle corresponding to $\mathcal{V}^c$. In order to prove this lemma we will use the table given in [67] §5, which gives numerical relations between eigenvalues of monodromies of a local system of a complex variation, eigenvalues of the corresponding Higgs field and jumps of filtrations on the corresponding filtered regular Higgs bundle and filtered local system.

1. Since $\Theta = \sum_{p+q=k} \Theta^{p,q}$, where $\Theta^{p,q} : \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \omega_Y(S)$, one gets that the $\mathcal{O}_Y$-linear operator $\Theta : \mathcal{H} \to \mathcal{H} \otimes \omega_Y(S)$ is nilpotent. Following the notations from the table we have
$$b + ci = 0,$$
i.e. the jumps of the filtration on $\mathbb{V}_s$ are at $\beta = -2b = 0$, for all $s \in S$. Hence, $\deg \mathbb{V} = 0$, see Definition 2.27. Moreover, by Simpson's correspondence one gets $\deg \mathcal{H}_f = 0$.

2. The eigenvalues of monodromies around points of S of the local system$\mathbb{V}$ are $e^{-2\pi\alpha i + 4\pi c = 0}$. Assuming that monodromies are unipotent, one has
$$-2\pi\alpha i + 4\pi c = 0,$$
i.e. the jumps α of the filtered regular Higgs bundle $\mathcal{H}_f$ are zero at every $s \in S$ and by Remark 2.28 one has: $\deg \mathcal{H}_f = \deg \mathcal{H} = 0$. Hence, the Higgs bundle $\mathcal{H}$ is polystable of degree 0.

3. Using the previous notations, we have to prove that α and c are zero. From point 1) one gets that $c = 0$. Since $\deg \mathbb{V} = 0$, the corresponding filtered Higgs bundle has degree zero, i.e. $\deg \mathcal{H}_f = 0$. The fact that $\deg \mathcal{H} = 0$ implies that all α's around points of S are zero, see Definition 2.24. Hence, monodromies around points of S of the local system $\mathbb{V}$ are unipotent.

$\square$

Natural examples of logarithmic Higgs bundles are the ones associated to geometric variations of projective families of varieties over a curve with singular fibers.

So, let $g : W \to Z$ be a projective family of n-varieties over a curve Z, with discriminant locus a finite set P. We will suppose that the divisor $\Delta = g^*(P)$ is a normal crossing divisor. The monodromy operators are
$$T = \rho([\alpha]),$$
where
$$\rho : \pi_1(Z \backslash P, z_0) \to \mathrm{GL}(H^n(W_{z_0}, \mathbb{C})), z_0 \in Z \backslash P,$$
is the corresponding representation for the local system $R^n g_* \mathbb{C}_{W \backslash \Delta}$ and $[\alpha]$ is the class of a loop around a point $p \in P$.

The monodromy theorem due to Alan Landman, states that the monodromy operators of families with singular fibers are at least quasiunipotent:

Theorem 2.13. *(Monodromy theorem, [47]) Let $f : W \to Z$ be a projective family, smooth over $U = Z \backslash P$, for P a finite set. Let k be an integer, then the local system $R^k g_* \mathbb{C}_{W \backslash g^{-1} P}$ has quasi unipotent monodromies.*

Hence, the monodromies around the points of P of the family $g : W \to Z$ are at least quasiunipotent. It is possible, after taking some smooth finite covering $\phi : Y \to Z$ ramified over P, to get a family $f : X \to Y$ with unipotent monodromies around points $S = \phi^* P$, where X is a desingularization of the normalization of the fiber product $W \times_Z Y$. The divisor $\Delta = f^*(S)$ is a normal crossing divisor. This is possible due to the semistable reduction theorem, which will be one of the topics in the following chapter, see Section 3.1. It is important to underline the fact that semistability of a family implies unipotent monodromies around the points of the discriminant locus of the family, see Theorem 3.8.

Now, let us describe the Higgs bundle associated to the above constructed projective family $f : X \to Y$, which has unipotent monodromies around the points of S. The construction of the Higgs bundle for the case of smooth families, which is done by using the relative complex $\Omega^\bullet_{X/Y}$, can be repeated here but this time using the complex $\Omega^\bullet_{X/Y}(\log \Delta)$. Almost all results that we state here are taken from the paper of N. Katz, [40] §1.

The Deligne extension of the flat bundle $R^n f_* \mathbb{C}_{X \setminus \Delta} \otimes \mathcal{O}_{Y \setminus S}$ is canonical, since the monodromies around points of S are unipotent. This extension is isomorphic to the bundle $E = R^n f_* \Omega^\bullet_{X/Y}(\log \Delta)$. The Gauss-Manin connection ∇ on the bundle $R^n f_* \mathbb{C}_{X \setminus \Delta} \otimes \mathcal{O}_{Y \setminus S}$ has at most logarithmic poles along Δ, i.e. it extends to a morphism:

$$\nabla_E : E \to E \otimes \Omega^1_X(\log \Delta), \tag{2.14}$$

which satisfies the Leibniz's rule. The filtration defined at (2.3) extends to a filtration $F^\bullet_E$ on $E = R^n f_* \Omega^\bullet_{X/Y}(\log \Delta)$ and one has:

$$\mathrm{Gr}^p(E) = F^p_E(E)/F^{p+1}_E(E) \cong R^q f_* \Omega^p_{X/Y}(\log \Delta). \tag{2.15}$$

The connection ∇_E satisfies the Griffiths's transversality property with respect to the filtration $F^\bullet_E$.

Hence, the logarithmic Higgs bundle associated to the local system $R^n f_* \mathbb{C}_{X \setminus \Delta}$ is defined by:

$$\mathcal{H} = \bigoplus_{p+q=n} R^q f_* \Omega^p_{X/Y}(\log \Delta),$$

with Higgs field $\Theta : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_Y(\log S)$, induced by ∇_E, and as we saw before (in the smooth case) it corresponds to the sum of edge morphisms

$$\Theta_{p,q} : R^q f_* \Omega^p_{X/Y}(\log \Delta) \to R^{q+1} f_* (\Omega^{p-1}_{X/Y}(\log \Delta)) \otimes \Omega^1_Y(\log S), \quad p + q = n, \tag{2.16}$$

of the long exact sequences obtained by taking f_* of the exact sequences

$$0 \to \Omega^{p-1}_{X/Y}(\log \Delta) \otimes f^* \Omega^1_Y(\log S) \to \Omega^p_X(\log \Delta) \to \Omega^p_{X/Y}(\log \Delta) \to 0.$$

Lemma 2.14. *In the previous notations, the Higgs bundle $\mathcal{H}$ has degree* 0 *and it is polystable.*

Proof. The monodromies around the points of S are unipotent, hence this result is a consequence of Lemma 2.12. $\square$

2.4 The Kodaira-Spencer map

We introduce now the Kodaira-Spencer map, which measures how a fiber in a family deforms in a small neighbourhood. The most important goal of this section is to show that this map induces the Higgs field of the Higgs bundle associated to a family. Here, we will explain the geometric Kodaira-Spencer map, but also we will define the algebraic prototype of this map. The theoretical background for this part is provided by [3] and [42].

Let $f : X \to Y$ be a family of Kähler varieties over a curve Y, with discriminant locus S with unipotent monodromies. We assume $\Delta = f^*(S)$ to be a normal crossings divisor. Recall that we have the exact sequence (see Section 1.2):

$$0 \to f^*\Omega^1_Y(\log S) \to \Omega^1_X(\log \Delta) \to \Omega^1_{X/Y}(\log \Delta) \to 0. \tag{2.17}$$

The extension of $f^*\Omega^1_Y(\log S)$ by the sheaf $\Omega^1_{X/Y}(\log \Delta)$ is given by a class

$$c \in \operatorname{Ext}^1\left(\Omega^1_{X/Y}(\log \Delta), f^*\Omega^1_Y(\log S)\right).$$

By some basic properties of the functor Ext, we have:

$$\begin{aligned}\operatorname{Ext}^1(\Omega^1_{X/Y}(\log \Delta), f^*\Omega^1_Y(\log S)) &= H^1\left(X, Hom_{\mathcal{O}_X}(\Omega^1_{X/Y}(\log \Delta), f^*\Omega^1_Y(\log S)\right)\\ &= H^1\left(X, T_{X/Y}(-\log \Delta) \otimes f^*\Omega^1_Y(\log S)\right),\end{aligned}$$

where $T_{X/Y}(-\log \Delta)$ is the dual bundle to the bundle $\Omega^1_{X/Y}(\log \Delta)$. Using the canonical morphism :

$$\alpha : H^1\left(X, T_{X/Y}(-\log \Delta) \otimes f^*\Omega^1_Y(\log S)\right) \to R^1 f_*\left(T_{X/Y}(-\log \Delta) \otimes f^*\Omega^1_Y(\log S)\right),$$

or

$$\alpha : H^1\left(X, T_{X/Y}(-\log \Delta) \otimes f^*\Omega^1_Y(\log S)\right) \to \Omega^1_Y(\log S) \otimes R^1 f_*\left(T_{X/Y}(-\log \Delta)\right),$$

we get :

$$\alpha(c) = \rho \in Hom_{\mathcal{O}_Y}\left((\Omega^1_Y(\log S))^{-1}, R^1 f_*(T_{X/Y}(-\log \Delta))\right),$$

or

$$\alpha(c) = \rho \in Hom_{\mathcal{O}_Y}\left(\mathcal{O}_X, R^1 f_*(T_{X/Y}(-\log \Delta)) \otimes \Omega^1_Y(\log S)\right).$$

Definition 2.29. The image of c by the canonical map α:

$$\rho : T_Y(-\log S) \to R^1 f_* \left(T_{X/Y}(-\log \Delta)\right),$$

is called the Kodaira-Spencer morphism of the family $f : X \to Y$.
The fiber map $\rho_y : T_{Y,y} \to H^1(X_y, T_{X_y})$ is the Kodaira-Spencer map at $y \in Y \backslash S$.

Remark 2.30. Note that the Kodaira-Spencer map can be seen as an element of

$$H^0\left(Y, \Omega_Y(\log S) \otimes R^1 f_*(T_{X/Y}(-\log \Delta))\right).$$

Let us recall some algebraic results from [42] §1 which we need in order to explicitly define the relation between the Kodaira-Spencer map of the family $f : X \to Y$ and the Higgs field of the corresponding Higgs bundle associated to the geometric variation of the family $f : X \to Y$.

Let $f : X \to Y$ be a map between smooth complex varieties. Let $\mathcal{H}$ and $\mathcal{F}$ be locally free sheaves on X and let $\mathcal{G}$ be a locally free sheaf on Y, such that we have a short exact sequence:

$$0 \to f^*\mathcal{G} \to \mathcal{H} \to \mathcal{F} \to 0.$$

Let $c \in \mathrm{Ext}^1(\mathcal{F}, f^*\mathcal{G})$ be the corresponding extension class. Then, we have the canonical morphism:

$$\alpha : \mathrm{Ext}^1(\mathcal{F}, f^*\mathcal{G}) = H^1(X, Hom(\mathcal{F}, f^*\mathcal{G})) \to R^1 f_*(Hom(\mathcal{F}, f^*\mathcal{G})) = \mathcal{G} \otimes R^1 f_*(\mathcal{F}^\vee).$$

The object $\alpha(c) \in Hom(\mathcal{O}_Y, \mathcal{G} \otimes R^1 f_*(\mathcal{F}^\vee))$ is called the algebraic prototype of the Kodaira-Spencer map of the previous exact sequence. The Koszul filtration induced by the previous exact sequence yields the exact sequences :

$$0 \to f^*\mathcal{G} \otimes \wedge^{p-1}\mathcal{F} \to (F^0/F^2)^p \to \wedge^p \mathcal{F} \to 0.$$

Moreover by Theorem 1.4.21 from [42], the edge morphism of the long exact sequence obtained by taking f_*, in degree $q \geq 0$:

$$\delta_{p,q} : R^q f_*(\wedge^p \mathcal{F}) \to R^{q+1} f_*(f^*\mathcal{G} \otimes \wedge^{p-1}\mathcal{F})$$

can be seen as the cup product with $\alpha(c)$, i.e. it can be factorised by the sequence of maps:

$$\begin{aligned}\delta_{p,q} : R^q f_*(\wedge^p \mathcal{F}) \cong \mathcal{O}_Y \otimes R^q f_*(\wedge^p \mathcal{F}) &\to \mathcal{G} \otimes R^1 f_*(\mathcal{F}^\vee) \otimes R^q f_*(\wedge^p \mathcal{F}) \\ &\to R^{q+1} f_*(f^*\mathcal{G} \otimes \wedge^{p-1}\mathcal{F}),\end{aligned}$$

where the first map is the map $\alpha(c) \otimes \mathrm{id}$ and the second map is the cup product morphism provided by the canonical morphism:

$$\mathcal{G} \to f_*(f^*)\mathcal{G} = R^0 f_*(f^*)\mathcal{G}.$$

As we saw in the previous section, the Higgs field of the Higgs bundle associated to a geometric variation of Hodge structures of weight k of a family of n-varieties is a sum of edge morphisms:

$$\Theta_{p,q} : R^q f_* \Omega^p_{X/Y}(\log \Delta) \to R^{q+1} f_*(\Omega^{p-1}_{X/Y}(\log \Delta)) \otimes \Omega^1_Y(\log S), p+q=k$$

of the long exact sequence obtained by taking f_* of the exact sequence:

$$0 \to f^* \Omega^1_Y(\log S) \otimes \Omega^{p-1}_{X/Y}(\log \Delta) \to \Omega^p_X(\log \Delta) \to \Omega^p_{X/Y}(\log \Delta) \to 0.$$

Hence, by the previous algebraic facts, one has that the (p,q)-part of the Higgs field Θ, i.e. the morphism $\Theta_{p,q}$, can be seen as the cup product with the Kodaira-Spencer map of the family $f : X \to Y$.

2.5 Period domains and period maps

In this section we will explain some basic facts about period domains, i.e. the domains which parametrize Hodge filtrations with fixed Hodge numbers on a given complex vector space. Later, we will introduce the period map of a given complex variation of Hodge structures.

Let $V_{\mathbb{Z}}$ be a free abelian group of finite rank and let $V = V_{\mathbb{Z}} \otimes \mathbb{C}$ be a complex vector space of dimension n. Let $\{h^{i,j}\}_{i+j=k}$ be a family of positive integers, where k is a fixed number, such that:

$$\sum_{i+j=k} h^{i,j} = n \quad \text{and} \quad h^{i,j} = h^{j,i}.$$

Let Q be a hermitian form on V, skew for k odd or symmetric for k even, integral and non-degenerate on $V_{\mathbb{Z}}$. Now, we define the numbers:

$$f^p = \sum_{i \geq p, i+j=k} h^{i,j}.$$

We want to describe the space which parametrizes the set of all filtrations $F^\bullet$ such that dim $F^p = f^p$ and F^p is Q-orthogonal to F^{q+1}, where $p+q = k$. Moreover, we ask the same questions with the additional condition, that $Q(\sqrt{-1}^{p-q}\xi, \xi) > 0$ for any nonzero vector $\xi \in F^p(V) \cap \overline{F^q(V)}$, i.e. the billinear form Q satisfies both Hodge-Riemann relations.

Recalling the definition of polarized Hodge structures, the second question can be reformulated as: what is the space which parametrizes the polarized Hodge structures of weight k of the triple $(V_{\mathbb{Z}}, V, Q)$ with fixed Hodge numbers $\{h^{i,j}\}_{i+j=k}$?

The answer to the first question is a flag manifold of type:

$$(f^k, f^{k-1}, ..., f^{[\frac{k+1}{2}]}).$$

This is a submanifold of the manifold $\prod_p \mathrm{Gr}(f^p, V)$, where $\mathrm{Gr}(m, V)$ denotes the Grassmannnian of subspaces of dimension m of V. We denote this flag manifold by $\check{D}$.

Let us answer to the second question with Proposition 4.3.3 from [10]:

Proposition 2.15. *The period domain D classifying the Hodge filtrations $F^\bullet$ of fixed dimension $f^p = \dim F^p = \sum_{i\geq p, i+j=k} h^{i,j}$ with fixed Hodge numbers $h^{i,j}$, satisfying both Hodge-Riemann relations is a domain of $\check{D}$ and it is a homogeneous manifold.*

A classical example of a period domain is the period domain of the weight-one variations of Hodge structures on a complex vector space of dimension $2g$. In this case the period domain D can be realized as the Siegel's upper half space, i.e.

$$\mathcal{H}_g = \{A \in \mathrm{GL}(g, \mathbb{C}) | A = A^t, \mathrm{Im} A > 0\}.$$

Let $\{\mathbb{V}, V, F^\bullet, \nabla, Q\}$ be a polarized complex variation of Hodge structures of weight k on a smooth curve Y. We are wondering whether this complex variation of Hodge structures gives rise to a natural map $\phi : Y \to D$, where D is the period domain for weight k Hodge structures. As $\mathbb{V}$ is a local system, it is not necessarily a constant sheaf and we do not have immediately a map $\phi : Y \to D$. But, let us argue locally and suppose that U is a small neighborhood of $y_0 \in Y$ such that the local system $\mathbb{V}$ restricted on U is a constant sheaf.

We trivialize the bundle $V = \mathbb{V} \otimes \mathcal{O}_Y$ on U by means of flat sections. In other words we can find a local frame of flat sections $\{\sigma_i\}$ such that $\{\sigma_i(y_0)\}$ is a basis of $\mathbb{V}_{y_0} = \bigoplus_{p+q=k} \mathbb{V}^{p,q}_{y_0}$ and $\{\sigma_i(y_0)\}_{f_{p+1} < i \leq f_p}$ is a basis of $\mathbb{V}^{p,q}_{y_0}$. Hence, we get a basis for $F^p(\mathbb{V}_{y_0})$ which is $\{\sigma_i(y_0)\}_{i \leq f_p}$. This yields a map:

$$\phi : U \to D,$$

defined as

$$\phi(y) = (F^k(\mathbb{V}_y), ..., F^{[\frac{k+1}{2}]}(\mathbb{V}_y)).$$

In order to have this map globally defined, let us take the universal covering $\tilde{Y} \to Y$ of the curve Y. The pullback of $\{\mathbb{V}, V, F^\bullet, \nabla, Q\}$ to $\tilde{Y}$ is a polarized complex variation of Hodge structures. The pullback of the local system $\mathbb{V}$ to $\tilde{Y}$ is a constant sheaf $\tilde{\mathbb{V}}$, since $\tilde{Y}$ is simply connected. Now, we choose the trivialization of $\mathbb{V}$ on $\tilde{Y}$ by flat sections as above and then the induced filtration $\tilde{F}^\bullet$ on $\tilde{Y}$ yields a morphism:

$$\tilde{\phi} : \tilde{Y} \to D.$$

Let ρ be the monodromy representation of the local system $\mathbb{V}$:

$$\rho : \pi_1(Y, y_0) \to \mathrm{Aut}(\mathbb{V}_{y_0}).$$

Let $\Gamma = \rho(\pi_1(Y, y_0))$ be the monodromy group of the local system $\mathbb{V}$. Since the complex variation is polarized, the monodromy group is discrete (see [10] p.156). The map $\tilde{\phi}$ is equivariant with respect to the $\pi_1(Y, y_0)$-action on $\tilde{Y}$ and the Γ-action on D. By this we mean:

$$\tilde{\phi}(\gamma \cdot \tilde{y}) = \rho(\gamma)\tilde{\phi}(\tilde{y}),$$

for $\gamma \in \pi_1(Y, y_0)$ and $\tilde{y} \in \tilde{Y}$. We define the period map of a polarized complex variation of Hodge structures $\{\mathbb{V}, V, F^\bullet, \nabla\}$, to be the induced map:

$$\phi : Y \to D/\Gamma.$$

This map is locally liftable, in the sense that for every $y \in Y$ the map ϕ restricted to some small open set U around y is seen as the restriction to a small open of the map $\tilde{\phi} : \tilde{Y} \to D$ followed by the quotient map $D \to D/\Gamma$.

Let us consider now a family $f : X \to Y$ of projective n-varieties over a curve Y. Then the period map of the corresponding geometric variation of Hodge structures of weight k of the family is holomorphic. This is a well known theorem of Griffiths, see [81] Theorem 10.21. The monodromy group Γ of the local system $R^k f_* \mathbb{C}_X$ is discrete and it acts properly discontinuously on D, hence the quotient D/Γ is a Hausdorff space.

The derivative of this period map at some point $y_0 \in Y$ is defined as the sum of linear morphisms which are the cup products with the Kodaira-Spencer class of this family at the point y_0. In fact by Lemma 5.3.3 from [10], the derivative of the period map at point $y_0 \in Y$ will be identified with the sum of (p, q)-parts:

$$(\Theta_{p,q})_{y_0}(v) : H^q(X_{y_0}, \Omega^p_{X_{y_0}}) \to H^{q+1}(X_{y_0}, \Omega^{p-1}_{X_{y_0}}), \quad v \in T_{Y,y_0}$$

induced by the Higgs field $\Theta = \sum_{p+q=k} \Theta_{p,q}$ of the Higgs bundle which arises from the geometric variation of Hodge structures of the family.

2.6 Connection with Teichmüller curves

In this section we will explain the connection between complex variations of Hodge structures of weight 1 and rank 2, and Teichmüller curves, which are geodesics in the moduli space of curves for a metric on that space induced by the Teichmüller metric on the Teichmüller space. We will finish this section by giving the proof of Möller's Theorem 5.3 from [52].

2.6.1 Teichmüller space

Before we give the proof of the main theorem which describes this relation, let us recall some well known facts about Teichmüller space. We will give a short review of a construction of this space and then we will define Teichmüller curves. The constructions explained here are from [7] and [36].

By the uniformization theorem, every compact Riemann surface of genus $g \geq 2$ can be represented as a quotient of the upper half plane $\mathbb{H}$ by a group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$, which is the representation of the fundamental group of the surface in $\mathrm{PSL}_2(\mathbb{R})$. Let us fix a compact Riemann surface X with fundamental group $\pi_1(X)$ and genus g. The fundamental groups of compact Riemann surfaces of genus g are isomorphic to a group G with $2g$ generators $\{\alpha_i, \beta_i\}_{i=1,\dots,g}$ which satisfy the condition:

$$\Pi_{i=1}^{g} \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = \mathrm{id},$$

where id is the identity element of the group. Let us fix an isomorphism $\pi_1(X) \cong G$. Such an isomorphism is called a marking .

A representation of G to $\mathrm{PSL}_2(\mathbb{R})$ means finding $2g$ elements $\{A_i, B_i\}_{i=1,\dots,g}$ from $\mathrm{PSL}_2(\mathbb{R})$ (or real matrices, up to the sign) which satisfy the condition:

$$\Pi_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = \mathrm{Id},$$

where Id is the identity element of the group $\mathrm{PSL}_2(\mathbb{R})$. We consider now the set of all representations $\rho : G \to \mathrm{PSL}_2(\mathbb{R})$,

$$\mathcal{R} = \{2g\text{-tuples of matrices from } \mathrm{PSL}_2(\mathbb{R}) : (A_i, B_i)_{i=1,\dots,g} | \Pi_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = \mathrm{Id}\}.$$

The asked condition for these matrices gives algebraic equations for entries of these matrices which determine the space of all representations of G to $\mathrm{PSL}_2(\mathbb{R})$.

From now on in this section, we suppose that X is a fixed Riemann surface of genus $g \geq 2$ and $\pi_1(X) \cong G$ is a fixed marking.

Definition 2.31. A subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ which acts on $\mathbb{H}$ in such a way that the quotient $\mathbb{H}/\Gamma$ is a Riemann surface is called a Fuchsian group.

Definition 2.32. The subset of all Fuchsian representations ρ of the group G, such that $\mathbb{H}/\rho(G)$ is a compact Riemann surface, in the set $\mathcal{R}$ modulo the action of $\mathrm{PSL}_2(\mathbb{R})$, which simultaneously conjugates all $2g$ matrices, is called the Teichmüller space of curves of genus g, or shorter, the Teichmüller space of genus g. We denote it as $\mathcal{T}_g$.

The Teichmüller space $\mathcal{T}_g$ has a natural topology induced by the matrix topology of the group $\mathrm{PSL}_2(\mathbb{R})$. Moreover, it has the structure of a complex manifold (see §6.1 [36]) and as such, it is biholomorphic to a bounded simply connected domain in $\mathbb{C}^{3g-3}$, see Theorem 6.6 from [36].

Now, we will introduce another definition of the Teichmüller space of curves of genus g. In order to do that we give a definition of quasiconformal maps.

Definition 2.33. A map $f : D \to D'$, where D and D' are two domains in $\mathbb{C}$ is said to be quasiconformal if it is an orientation preserving diffeomorphism and it satisfies:

$$K_f = \sup_{z \in D} \frac{1 + |\mu_f|}{1 - |\mu_f|} < \infty,$$

where $\mu_f = \frac{\frac{\partial f(z)}{\partial \bar{z}}}{\frac{\partial f(z)}{\partial z}}$ is called the Beltrami coefficient of map f and K_f is called the maximal dilatation of f. We say that f is a K_f-quasiconformal mapping.

Now, let us consider pairs (S, f) where:

- S is a closed Riemann surface of genus g;
- f is a quasiconformal map between the fixed surface X (from the beginning) and S.

We have the following definition:

Definition 2.34. We say that two pairs (S_1, f_1) and (S_2, f_2) are equivalent if $f_2 \circ f_1^{-1}$ is homotopic to a conformal map of S_1 to S_2. Let denote by $s = [S, f]$ the equivalence class of (S, f). The set of all such equivalence classes is called the Teichmüller space of the surface X, denoted by $\mathcal{T}_g(X)$ and (X, id_X) is called the base point of $\mathcal{T}_g(X)$.

The topology on the set $\mathcal{T}_g(X)$ is defined by the Teichmüller distance. Let us give the definition of Teichmüller distance:

Let $s_1 = [S_1, f_1]$ and $s_2 = [S_2, f_2]$ be two points in $\mathcal{T}_g(X)$. We define:

$\mathcal{Q}_{f_1,f_2} = \left\{ f : S_1 \to S_2 \mid f \text{ is a quasiconformal map homotopic to } f_2 \circ f_1^{-1} \right\}.$

Then, one has:

Definition 2.35. ([36] §5.1.3) For two points $s_1, s_2 \in \mathcal{T}_g(X)$ the Teichmüller distance is defined as:

$$d_T(s_1, s_2) = \inf_{f \in \mathcal{Q}_{f_1,f_2}} \log K_f,$$

where K_f is the maximal dilatation of f.

It is important to note that the Teichmüller space $\mathcal{T}_g(X)$ of the surface X with the Teichmüller distance is a complete space, by Theorem 5.4 [36]. $\mathcal{T}_g(X)$ does not depend on the chosen base point X and is identified with $\mathcal{T}_g$, the Teichmüller space of curves of genus g, see Proposition 5.4 and Remark 3 from [36] §5. From now on we consider that $\mathcal{T}_g$ is endowed with the Teichmüller distance d_T.

Definition 2.36. The moduli space $\mathcal{M}_g$ of curves of genus g is defined to be the set of all biholomorphic equivalence classes $[S]$ of closed Riemann surfaces S of genus g.

It turns out that $\mathcal{M}_g$ is naturally an algebraic singular variety (when $g \geq 4$ the singular points are classes $[S]$ where S is a compact Riemann surface such that the group of automorphisms of that surface is not trivial, i.e. $\mathrm{Aut}(S) \neq \{\mathrm{Id}\}$, see [50]).

Definition 2.37. The mapping class group MCG_g of genus g (or the Teichmüller modular group) is the group of all orientation-preserving diffeomorphisms of the surface X modulo the group of those diffeomorphisms which are homotopic to the identity map on that surface.

The group MCG_g acts properly discontinuously on $\mathcal{T}_g$ as a discrete subgroup of the group of biholomorphic automorphisms of $\mathcal{T}_g$, see Theorem 6.18 from [36]. The quotient $\mathcal{T}_g/\mathrm{MCG}_g$ is identified with $\mathcal{M}_g$. In this way $\mathcal{M}_g$ inherits the complex structure from $\mathcal{T}_g$, but as an orbifold.

Theorem 2.16. *(6.19 [36]) The moduli space $\mathcal{M}_g$, of curves of genus $g \geq 2$, has a structure of normal complex analytic space of dimension $3g-3$.*

The Teichmüller distance on $\mathcal{T}_g$ is invariant under the action of the mapping class group MCG_g and it induces a metric on the moduli space of curves of genus g.

Definition 2.38. The complex geodesics for the Teichmüller distance are called *Teichmüller disks.* A complex submanifold of Teichmüller space is called totally geodesic if it contains a Teichmüller disk trough any of its two points. A complex subvariety of the moduli space of curves of genus g is called totally geodesic if the components of its preimage in the Teichmüller space are totally geodesic.

In quite a few cases Teichmüller disks descend to algebraically defined curves in $\mathcal{M}_g$, which are called Teichmüller curves.

Definition 2.39. A Teichmüller curve $C \to \mathcal{M}_g$ is an algebraic curve in the moduli space of curves which is a complex geodesic with respect to the metric which is induced by the Teichmüller metric on $\mathcal{M}_g$.

As we already said the Teichmüller space is a complex space, therefore besides Teichmüller distance which we defined above, there is one other natural metric on that space. That is the Kobayashi metric, which is induced by the Poincaré metric on the upper half plane. Let us define this metric:

Definition 2.40. ([36] §6.4) Let d be the Poincaré metric on the upper half plane $\mathbb{H}$. Let W be a complex manifold and let $x, y \in W$ be two points. We define:

$$d_K^1(x,y) = \inf_{\psi} d(a,b),$$

where the infimum is taken over all points $a, b \in \mathbb{H}$ such that there exists a holomorphic mapping $\psi : \mathbb{H} \to W$ with $\psi(a) = x, \psi(b) = y$. For any positive integer n, we put:

$$d_K^n(x,y) = \inf \sum_{i=1}^{n} d_K^1(a_{i-1}, a_i),$$

where the infimum is taken over all points $a_0, ..., a_n \in W$ with $a_0 = x$ and $a_n = y$. For all positive integers n, we have:

$$d_K^{n+1}(x,y) \leq d_K^n(x,y), \quad x,y \in W.$$

The Kobayashi metric is defined as:

$$d_K(x,y) = \lim_{n\to\infty} d_K^n(x,y).$$

A theorem of Royden [61] states that the Kobayashi metric and the Teichmüller distance coincide on Teichmüller space, see Theorem 6.21 in [36].

2.6.2 "Good" metrics

Let us explain the notion of "good" metrics, due to Mumford. Following Peters's paper [57], we will use these metrics in order to compute the degree of (p,q)-parts of a Higgs bundle which corresponds to a complex variation of weight $p+q=k$, on a punctured projective smooth curve. We will explain what is a good metric on a line bundle on a punctured projective smooth curve and several results which follow from imposing this metric. After, we will see that a Hodge metric induced by the polarization of a complex variation of Hodge structures induces a good metric on (p,q)-parts of the corresponding Higgs bundle, which provides us a method to compute the degree of (p,q)-parts.

Definition 2.41. ([57] §3) Let Y be a smooth projective curve and let S a finite set of points on Y. We set $U = Y \backslash S$. Let $\mathcal{L}$ be a holomorphic line bundle on Y. We denote by $\mathcal{L}_U := \mathcal{L}_{|U}$, the restriction of $\mathcal{L}$ on U. A hermitian metric h on the line bundle $\mathcal{L}_U$ is said to be "good" at point $s \in S$ if for some coordinate neighborhood (Δ, t) centered at s and for all generating holomorphic sections $\sigma \in H^0(\Delta, \mathcal{L})$ the following bounds are valid:

$$C_1\left(\log(\frac{1}{|t|})\right)^{-\beta} \leq h\left(\sigma(t),\sigma(t)\right) \leq C_2\left(\log(\frac{1}{|t|})\right)^{\beta},$$

$$\left|\frac{\partial \log h(\sigma(t),\sigma(t))}{\partial t}\right| \leq C_3|t|^{-1}\left(\log\frac{1}{|t|}\right)^{-1},$$

and

$$\left|\frac{\partial^2 \log h(\sigma(t),\sigma(t))}{\partial t\partial \bar{t}}\right| \leq C_4|t|^{-2}\left(\log\frac{1}{|t|}\right)^{-2},$$

for C_1, C_2, C_3, C_4 positive constants and β a non negative integer.

In the case when the metric h is good at all points $s \in S$, one has the canonical extension $\mathcal{L}$ of $\mathcal{L}_U$ to Y determined by the metric. Let us explain this. If we have a holomorphic line bundle $\mathcal{L}$ on Y and its restriction $\mathcal{L}_U$ has a good hermitian metric near the punctures $s \in S$ then $\mathcal{L}$ is not an arbitrary extension of $\mathcal{L}_U$ to Y, it is a canonical extension determined by the first growth condition from the definition. It is due to the fact that the extension $\mathcal{L}$ of $\mathcal{L}_U$ to Y is defined by:

$$H^0(\Delta, \mathcal{L}) = \left\{ \sigma \in H^0(\Delta \cap U, \mathcal{L}_U) | h\left(\sigma(t), \sigma(t)\right) \leq C(\log(\frac{1}{|t|}))^{\beta} \right\},$$

where $C > 0$ and β depends on s.

In the case of good metric the second condition from the definition, as it is indicated in [57] §3, implies that the Chern form $\frac{1}{2i\pi}\partial\overline{\partial}\log h$ is integrable , while the third condition implies that the degree of $\mathcal{L}$ is:

$$\begin{aligned} \deg \mathcal{L} &= \frac{1}{2i\pi}\int_U \partial\overline{\partial}\log h + \sum_{s\in S}\alpha_s \\ &= \frac{i}{2\pi}\int_U \Theta_h + \sum_{s\in S}\alpha_s, \end{aligned}$$

where Θ_h is the curvature form of h and α_s's are residues of the metric h at $s \in S$. The residues for the "good" metric h are defined as:

$$\alpha_s = \lim_{\epsilon\to 0}\frac{1}{2i\pi}\int_{|t|=\epsilon} \overline{\partial}\log h_\epsilon(\sigma(t), (t)),$$

where h_ϵ is a smoothening of h, which coincides with h on the annulus:

$$\frac{2}{3}\epsilon < |t| < \frac{4}{3}\epsilon.$$

Moreover, if we recall the definition of a filtered vector bundle, one can see that these residues correspond to the jumps of the filtration on the vector bundle in the punctures.

Now, we suppose that $\{\mathbb{H}, H_U, F^\bullet, Q\}$ is a polarized complex variation of Hodge structures of weight k on a punctured projective smooth curve $U = Y \backslash S$, with unipotent monodromies. Let:

$$\mathcal{H}_U = \bigoplus_{p+q=k}\mathcal{H}_U^{p,q},$$

be the corresponding Higgs bundle. The polarization Q induces the Hodge metric h on $\mathcal{H}_U$ and the induced metrics on $\mathcal{H}_U^{p,q}$ denoted by $h_{p,q}$. Let:

$$\mathcal{H} = \bigoplus_{p+q=k}\mathcal{H}^{p,q},$$

be the logarithmic Higgs bundle associated to the canonical extension H of H_U. By Proposition 2.2.1 from [57] the Hodge metrics h and $h_{p,q}$ are good metrics at all points $s \in S$. So, we get:

$$\deg(\det \mathcal{H}^{p,q}) = \frac{i}{2\pi} \int_U \mathrm{trace}(\Theta_{h_{p,q}}) + \sum_{s\in S} \alpha_s.$$

Moreover, in the case of unipotent monodromies all α_s's are zero and:

$$\deg(\det \mathcal{H}^{p,q}) = \frac{i}{2\pi} \int_U \mathrm{trace}(\Theta_{h_{p,q}}).$$

2.6.3 Complex variations of weight 1 and rank 2

Let us give a few more details about complex variations of Hodge structures of weight 1 and rank 2, in order to describe their strong relation with Teichmüller curves.

Let $\mathcal{L}^c = \{\mathbb{L}_U, L_U, \nabla, Q\}$ be a polarized complex variation of Hodge structures of weight 1 and rank 2 on a curve $U = Y \backslash S$, where Y is a smooth projective curve and S is a finite set of points on Y. We will assume the monodromies around points of S to be unipotent. Moreover, we will assume that $\chi(U) < 0$. Then, the universal covering of U is the upper half plane:

$$\tilde{U} = \mathbb{H}.$$

Since the rank of $\mathcal{L}^c$ is 2, by section 2.5 the period domain for this variation is the upper half plane $\mathbb{H}$. The lifting of the period map is:

$$\phi : \tilde{U} \to \mathbb{H}.$$

Since the variation is polarized, there is an integral locally free sheaf $\mathbb{L}_{\mathbb{Z}}$ such that $\mathbb{L}_U = \mathbb{L}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ and the polarization Q takes integral values on $\mathbb{L}_{\mathbb{Z}}$.

Let $\rho : \pi_1(U, *) \to \mathrm{SL}_2(\mathbb{C})$ be the representation of the local system $\mathbb{L}_U$. Let $\Gamma = \mathrm{Im}(\rho)$ be the monodromy group of the representation ρ. It is well known, see [10] p.156, that the monodromy group of a polarized complex variation of Hodge structures is discrete. Hence, we have $\Gamma \subset \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{R})$ and

$$\rho : \pi_1(U, *) \to \mathrm{SL}_2(\mathbb{R}).$$

The lifting of the period map ϕ is equivariant with respect to the $\pi_1(U, *)$-action on $\tilde{U}$ and the Γ-action on $\mathbb{H}$. The group Γ can be also considered as a lattice in $\mathrm{PSL}_2(\mathbb{R})$.

Definition 2.42. In the previous notations, the representation ρ is called a uniformization representation if one has a biholomorphism:

$$U \cong \mathbb{H}/\Gamma.$$

Let $\mathcal{E} = \mathbb{H} \times \mathbb{C}^2$ be the trivial bundle on $\mathbb{H}$. Then, one has the exact sequence of bundles on $\mathbb{H}$:

$$0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{Q} \to 0,$$

where the subbundle $\mathcal{S}$ of $\mathbb{H} \times \mathbb{C}^2$ is the tautological bundle of $\mathbb{P}^1_{\mathbb{C}}$, restricted to $\mathbb{H}$. This holds since $\mathbb{H}$ can be seen as a domain of $\mathbb{P}^1_{\mathbb{C}}$. The bundle $\mathcal{Q}$ is the quotient bundle. There is a natural isomorphism, see [81] §10,

$$T_{\mathbb{H}} = \mathrm{Hom}(\mathcal{S}, \mathcal{Q}) \cong \mathcal{S}^{-1} \otimes \mathcal{Q}, \tag{2.18}$$

where $T_{\mathbb{H}}$ is the tangent bundle of $\mathbb{H}$.

Let H be the Hodge metric on $\mathcal{E} = \mathbb{H} \times \mathbb{C}^2$ which is provided by the bilinear form Q in the sense of Definition 2.10 and which is $\mathrm{SL}_2(\mathbb{R})$-invariant. Then we get the induced Hermitian metrics on the subbundle $\mathcal{S}$ and the quotient bundle $\mathcal{Q}$. We denote the corresponding curvatures as $\Theta_{\mathcal{S}}$ and $\Theta_{\mathcal{Q}}$. By (2.18) the Hodge metric H induces a $\mathrm{SL}_2(\mathbb{R})$-invariant metric on the holomorphic tangent bundle $T_{\mathbb{H}}$. This metric is a Kähler metric and it coincides with the Poincaré metric on $\mathbb{H}$, up to a constant, by Theorem 1.4 from [83]. We denote the associated Kähler form by ω_T and for the corresponding curvature Θ_T we have:

$$\Theta_T = -\Theta_{\mathcal{S}} + \Theta_{\mathcal{Q}}.$$

For ω_T suitably normalized, the corresponding curvature Θ_T, by §13.3 [10], satisfies:

$$-i\Theta_T = -\text{ Gaussian curvature} \cdot \omega_T.$$

The Gaussian curvature of the upper half plane with Poincaré metric is equal to -1, therefore:

$$\omega_T = -i\Theta_T. \tag{2.19}$$

The pullback of the sequence $0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{Q} \to 0$ to U is the sequence:

$$0 \to \phi^*\mathcal{S} \to \phi^*\mathcal{E} \to \phi^*\mathcal{Q} \to 0,$$

and these pullbacks are well defined on U since the map ϕ is equivariant with respect to the $\pi_1(U,*)$-action on $\tilde{U}$. One has:

$$\det\, \phi^*\mathcal{E} = \det\, \phi^*\mathcal{S} \otimes \det\, \phi^*\mathcal{Q}.$$

Then using that $\phi^*\mathcal{S}$ and $\phi^*\mathcal{Q}$ are line bundles, and $\phi^*\mathcal{E}$ is a flat bundle we get:

$$\phi^*\Theta_{\mathcal{S}} = -\phi^*\Theta_{\mathcal{Q}}.$$

Let us denote $\phi^*\Theta_{\mathcal{S}} = \Theta$. The metric form ω_T is $\mathrm{SL}_2(\mathbb{R})$-invariant, and its pullback $\phi^*\omega_T$ to $\tilde{U}$ descends to U. Moreover, by (2.19) it satisfies:

$$\phi^*\omega_T = -i\phi^*\Theta_T = -i\phi^*(-\Theta_{\mathcal{S}} + \Theta_{\mathcal{Q}}) = 2i\Theta. \tag{2.20}$$

The flat bundle $\phi^*\mathcal{E}$ on U is the same as the flat bundle $L_U = \mathbb{L}_U \otimes \mathcal{O}_U$. Moreover, for the corresponding Higgs bundle on U by [81] §10 (p.250) we have $F^1(L_U) = \phi^*\mathcal{S}$ and

$$L_U^{1,0} = \mathrm{Gr}^1(L_U) = F^1(L_U)/F^2(L_U) = \phi^*\mathcal{S},$$

since $F^2(L_U) = 0$, and

$$L_U^{0,1} = \mathrm{Gr}^0(L_U) = F^0(L_U)/F^1(L_U) = \phi^*\mathcal{Q},$$

since $F^0(L_U) = L_U = \phi^*\mathcal{E}$.

Now, since the monodromies around points of S are unipotent one has the canonical Deligne extension of the flat bundle L_U to a flat bundle on Y, whose corresponding logarithmic Higgs bundle on Y is $L = L^{1,0} \oplus L^{0,1}$, with Higgs field $\Theta_L : L^{1,0} \to L^{0,1} \otimes \omega_Y(S)$. As we saw before, the induced Hodge metric on $L_U^{1,0} = \phi^*\mathcal{S}$ by the pullback of the Hodge metric H, is a "good" metric, which implies that the first Chern form is integrable. Then the degree of the Higgs term $L^{1,0}$ is:

$$\deg L^{1,0} = \frac{i}{2\pi}\int_U \Theta + \sum_{s\in S} \alpha_s,$$

where α_s's are residues of the induced Hermitian metric on $L^{1,0}$ at points of the set of punctures S. The unipotent monodromies imply $\alpha_s = 0, s \in S$ and one has:

$$\deg L^{1,0} = \frac{i}{2\pi}\int_U \Theta. \tag{2.21}$$

Let us recall the definition of Toledo invariant of the representation ρ. The map ϕ is a holomorphic map and hence it is a harmonic map, see [73] §4 Proposition 3.14. The monodromies around punctures are unipotent, which implies that the map ϕ is tame and hence it is of finite energy, see [45] Proposition 4.5. One can find the expression for the Toledo invariant in Proposition 4.6 from [45], but in the complex dimension 1 it coincides with the invariant defined in [72] and it is:

$$\tau(\rho) = \int_U \phi^*\omega_T. \tag{2.22}$$

Lemma 2.17. *In the previous notations, we have:*

$$\tau(\rho) = 4\pi \deg L^{1,0}.$$

Proof. This is a direct consequence of equations (2.20), (2.21) and the definition of the Toledo invariant. □

Let us recall Theorem 4.3 from [45], the result of Koziarz and Maubon about the Toledo invariant and uniformization representations of the fundamental group of a punctured complex curve.

Theorem 2.18. *Let Y be a complex compact smooth curve of genus g and let S be the set of punctures on Y. Assume that $\chi(Y\backslash S) = 2-2g-\#S < 0$. Let $U = Y\backslash S$ and let $\rho : \pi(U,*) \to \mathrm{PSL}_2(\mathbb{R})$ be a homomorphism. Then, one has:*

$$\tau(\rho) = -2\pi\chi(Y\backslash S)$$

if and only if ρ is a uniformization representation.

We end this part with Proposition 1.2 from [80] about the relation between the degree of the (1,0)-term of a Higgs bundle and its Higgs field being an isomorphism:

Lemma 2.19. *Let $\mathbb{L}$ be a rank 2 polystable local system of degree zero on $U = Y\backslash S$, where S is a finite set of points on the smooth projective curve Y such that $\chi(Y\backslash S) < 0$. Let $(L = L^{1,0} \oplus L^{0,1}, \Theta)$ be the logarithmic Higgs bundle corresponding to $\mathbb{L}$. Assume that $\Theta_{|L^{0,1}} = 0$ and $\Theta_{1,0} = \Theta_{|L^{1,0}} : L^{1,0} \to L^{0,1} \otimes \omega_Y(S)$. If*

$$2\deg(L^{1,0}) = \deg(\omega_Y(S))$$

then $\Theta_{1,0}$ is an isomorphism.

Proof. The degree of the logarithmic Higgs bundle L of rank two is zero and L is polystable. Since $\deg L^{1,0} > 0$, one gets that $L^{1,0}$ is not Θ-invariant and $\Theta_{1,0} \neq 0$. Moreover, $\deg L^{0,1} = -\deg L^{1,0} < 0$. This implies that $L^{1,0}$ and $L^{0,1}$ are of rank 1. Moreover, since $\Theta_{1,0} \neq 0$ one gets that $\Theta_{1,0}$ is injective as a morphism of sheaves.

Since $\Theta_{1,0}(L^{1,0}) \subseteq L^{0,1} \otimes \omega_Y(S)$, we get the Higgs subsheaf:

$$L' = (L^{1,0} \oplus \Theta_{1,0}(L^{1,0}) \otimes (\omega_Y(S))^{-1}, \Theta_{1,0})$$

of the Higgs bundle L. One has:

$$\deg L' = \deg L^{1,0} + \deg \Theta_{1,0}(L^{1,0}) - \deg(\omega_Y(S)) = 0,$$

since $\Theta_{1,0}(L^{1,0}) \cong L^{1,0}$. Since the Higgs bundle L is polystable of degree 0, L' is a direct factor of L. This is only possible when $\Theta_{1,0}(L^{1,0}) \otimes (\omega_Y(S))^{-1} = L^{0,1}$, which implies $\Theta_{1,0}$ an isomorphism. □

Now, we will state and prove Lemma 2.1 from [80]. Note that the proof is slightly different from the one given by Viehweg and Zuo. Here, we are combining the previous few results in order to prove that if the Higgs field of a complex variation of weight 1 rank 2 is an isomorphism, then the corresponding representation is uniformization.

Lemma 2.20. *Let $\mathcal{L}^c = \{\mathbb{L}, L, \nabla, Q\}$ be a polarized complex variation of Hodge structures of weight 1, rank 2, with a nontrivial Higgs field on a smooth punctured curve $U = Y\backslash S$. Let $\rho : \pi_1(U,*) \to \mathrm{PSL}_2(\mathbb{R})$ be the corresponding* *representation and let Γ denote the image of ρ. Assume that the local monodromies around the points $s \in S$ are unipotent. The corresponding Higgs field of $\mathcal{L}^c$ is an isomorphism if and only if:*

$$Y \backslash S \simeq \mathbb{H}/\Gamma.$$

Proof. Using that $\mathcal{L}^c$ is a complex variation of Hodge structure of weight one, we have the associated logarithmic Higgs bundle $L = L^{1,0} \oplus L^{0,1}$ on Y with non-trivial Higgs field:

$$\Theta : L^{1,0} \to L^{0,1} \otimes \omega_Y(S).$$

Since the monodromies around points of S are unipotent, the Higgs bundle L is polystable of degree 0. As $\Theta_{|L^{0,1}} = 0$, the subbundle $L^{0,1}$ is Θ-invariant. As L is polystable of degree 0, we get $\deg L^{0,1} \leq 0$. So, if $\deg L^{0,1} = 0$ then $\deg L^{1,0} = 0$. Polystability of L implies that $L^{1,0}$ and $L^{0,1}$ are stable Higgs bundles of degree 0 with trivial Higgs fields, in other words Θ is trivial. But, this is not the case here and $\deg L^{0,1} < 0$. Hence, one gets:

$$\deg L^{1,0} > 0.$$

Let us suppose that Θ is an isomorphism:

$$\Theta : L^{1,0} \otimes (L^{0,1})^{-1} \cong \omega_Y(S),$$

then the positivity of the degree of the left side implies:

$$\deg(\omega_Y(S)) = 2g - 2 + \#S > 0,$$

so we get

$$\chi(Y \backslash S) < 0.$$

By the uniformization theorem, the universal covering of $U = Y \backslash S$ is the upper half plane, i.e.

$$\tilde{U} = \mathbb{H}.$$

Let $\phi : \tilde{U} \to \mathbb{H}$ be a lifting of the period map. This map is equivariant with respect to the $\pi_1(U, *)$-action on $\tilde{U}$ and the Γ-action on $\mathbb{H}$. And we have the diagram :

$$\begin{tikzcd} \widetilde{U} \arrow[r, "\phi"] \arrow[d] & \mathbb{H} \arrow[d] \\ U \arrow[r] & \mathbb{H}/\Gamma \end{tikzcd}$$

Then using the formula $\tau(\rho) = 4\pi \deg L^{1,0}$ we get:

$$\tau(\rho) = 2\pi(2g - 2 + \#S).$$

By Theorem 2.18, one has:

$$U \cong \mathbb{H}/\Gamma,$$

and the map ϕ is an isomorphism.

Since the map ϕ is holomorphic, then by Claim 4.11 from [45] the map ϕ is an isometry for the Poincaré metric.

In the other direction, we suppose that $U \cong \mathbb{H}/\Gamma$, then Theorem 2.18 implies:

$$\tau(\rho) = 2\pi(2g - 2 + \#S),$$

and hence,

$$\deg L^{1,0} = \frac{1}{2}(2g - 2 + \#S).$$

The result follows by Lemma 2.19.

□

Remark 2.43. As a consequence of the previous lemma, when the Higgs field of a complex variation, of weight 1 and rank 2, is an isomorphism, then the corresponding representation is a uniformization and it is equivalent to the fact that the period map is a holomorphic isometry for the Poincaré metric on $\mathbb{H}$.

The last auxiliary lemma in this section will be one that explains the relation between holomorphic isometries and Teichmüller discs, i.e. a geodesic in the Teichmüller space for the Kobayashi metric. It can be found in the paper [22] as Corollary 1.

Lemma 2.21. *(§7, [22]) Let d be the Poincaré metric on the upper half plane and let d_K be the Kobayashi metric on the Teichmüller space $\mathcal{T}_g$ of genus $g \geq 2$. Let $i : \mathbb{H} \to \mathcal{T}_g$ be a holomorphic map. If one has that:*

$$d_K(i(z_1), i(z_2)) = d(z_1, z_2)$$

for any two different points $z_1, z_2 \in \mathbb{H}$, the image of $\mathbb{H}$ by i in $\mathcal{T}_g$ is a Teichmüller disc.

Let us finish this section by giving the proof of Theorem 5.3 from [52]. This theorem explains the nature of families of curves, of genus at least 2, whose geometric variation has a subvariation with maximal Higgs field. In fact, it states that the base curve of the family is a Teichmüller curve in the moduli space of curves of genus equal to the genus of smooth fibers of that family.

Theorem 2.22. *Let $f : X \to Y$ be a semistable family of curves of genus $g \geq 2$ over a smooth projective curve, smooth over $U = Y \backslash S$, where S is a finite set of points on Y, and let $V = f^{-1}(U)$. Suppose that $R^1 f_* \mathbb{C}_V$ contains a sub-variation of Hodge structures $\mathbb{L}$ of rank 2 whose Higgs field is an isomorphism. Then U is a Teichmüller curve.*

Proof. The family $f : X \to Y$ is semistable, which implies that $R^1 f_* \mathbb{C}_V$ has unipotent monodromies around the discriminant locus S, and hence the sub-variation $\mathbb{L}$ has unipotent monodromies around points of S (see Theorem 3.8). The Higgs field of the logarithmic Higgs bundle $L = L^{1,0} \oplus L^{0,1}$ corresponding to $\mathbb{L}$ is an isomorphism

$$\Theta_L : L^{1,0} \cong L^{0,1} \otimes \omega_Y(S),$$

therefore by Lemma 2.20 the universal covering of U is the upper-half plane, i.e. $\widetilde{U} = \mathbb{H}$, and moreover

$$U \cong \mathbb{H}/\Gamma,$$

where Γ is the image of the fundamental group $\pi_1(U, *)$ by the corresponding representation ρ of the local system $\mathbb{L}$. So, the representation ρ is Fuchsian. In particular, this means that the local system $\mathbb{L}$ is irreducible (see [52] p.11). Also, Θ_L being an isomorphism implies that the period map of $\mathbb{L}$ is an isomorphism, by Remark 2.43.

By Deligne's semi-simplicity theorem or by the modified version for complex variations of Hodge structures, Theorem 2.5, we have a decomposition of the local system $R^1 f_* \mathbb{C}_V$ given as:

$$R^1 f_* \mathbb{C}_V = \bigoplus_i (\mathbb{L}_i \otimes W_i),$$

where $\mathbb{L}_i$ are pairwise non-isomorphic irreducible $\mathbb{C}-$local systems and W_i are non-zero $\mathbb{C}-$vector spaces. So, since $\mathbb{L}$ is an irreducible sub-local system of $R^1 f_* \mathbb{C}_V$ we can suppose that in the previous decomposition $\mathbb{L}_1 = \mathbb{L}$.

Let $m : U \to \mathcal{M}_g$ be the moduli map, i.e. the map which sends a point $y \in U$ to the point in $\mathcal{M}_g$ corresponding to the fiber $f^{-1}(y)$. Let $\tilde{m} : \mathbb{H} \to \mathcal{T}_g$ be a lift of this map. This map is well defined by a choice of Teichmüller marking on some smooth fiber. Hence, we have the diagram:

$$\begin{tikzcd} \widetilde{U} \arrow[r, "\tilde{m}"] \arrow[d] & \mathcal{T}_g \arrow[d] \\ U \arrow[r, "m"] & \mathcal{M}_g \end{tikzcd}$$

One has $\dim(R^1 f_* \mathbb{C}_V)_y = \dim H^1(X_y, \mathbb{C}) = 2g$, then by Section 2.5 the lifting of the period map for the geometric variation of the family $f : X \to Y$ is:

$$\varphi : \tilde{U} \to \mathbb{H}_g.$$

We have supposed that $\mathbb{L} = \mathbb{L}_1$ which is a direct irreducible factor of the decomposition of the local system $R^1 f_* \mathbb{C}_V$. This is possible by choosing a suitable symplectic base of fibers of $R^1 f_* \mathbb{C}_V$ such that the first pair in that base spans fibers of $\mathbb{L}_1 = \mathbb{L}$. We define a map:

$$p_{11} : \mathbb{H}_g \to \mathbb{H},$$

which is just the projection of the matrix in $\mathbb{H}_g$ on its (1,1)-entry, which corresponds to the local system $\mathbb{L}$ in the period domain for $R^1 f_* \mathbb{C}_V$.

The map φ can be factorized as $\varphi = j \circ \tilde{m}$, where $j : \mathcal{T}_g \to \mathbb{H}_g$ is a lift of the Jacobian map, i.e. the map which associates to the curve its Jacobian (the period map). Moreover, it means that the map ϕ, the period map of $\mathbb{L}$, can be factorized as on the diagram:

$$\begin{tikzcd}
\mathcal{T}_g \arrow[r, "j"] & \mathbb{H}_g \arrow[d, "p_{11}"] \\
\widetilde{U} \arrow[u, "\tilde{m}"] \arrow[r, "\phi"] & \mathbb{H}
\end{tikzcd}$$

So we have: $\phi = p_{11} \circ j \circ \tilde{m}$. We will prove that $\tilde{m}$ is an isometry for the Kobayashi metric d_K. By the definition of the Kobayashi metric, it is plain that for $x, y \in U$ one has:

$$d(x, y) \geq d_K(\tilde{m}(x), \tilde{m}(y)),$$

where d is the Poincaré metric on the upper half plane.

Since ϕ is an isometry, we get: $d(x, y) = d(\phi(x), \phi(y))$. The map j is distance decreasing, by Corollary 13.4.3 from [10]. The projection p_{11} does not increase the metric. So, one has:

$$d(x, y) = d(\phi(x), \phi(y)) = d(p_{11} \circ j \circ \tilde{m}(x), p_{11} \circ j \circ \tilde{m}(y)) \leq d_K(\tilde{m}(x), \tilde{m}(y)).$$

Hence, $d(x, y) = d_K(\tilde{m}(x), \tilde{m}(y))$, and so $\tilde{m}$ is an isometry for the Kobayashi metric. Since $\tilde{m}$ is a holomorphic isometry for the Kobayashi metric, the image of U in $\mathcal{M}_g$ is an algebraic curve whose lift in $\mathcal{T}_g$ is a Teichmüller disc, hence the result follows by Lemma 2.21. □

Chapter 3

A theorem of Viehweg and Zuo

In this chapter, we will prove in detail Proposition 2.1 from [55], an Arakelov inequality for rank 1 subsheaves of the direct image of the relative pluricanonical sheaf of a semistable family of n-folds over a curve. We will focus on the case of semistable families of curves. Later, we discuss the case of equality in the Arakelov inequality, namely the maximal case.

3.1 Semistable families of curves

Let us recall the definitions of semistable curve, semistable map, weak semistable family and semistable family of curves and state the theorem of semistable reduction, which are important for the construction carried out in the proof of the theorem of Viehweg and Zuo. Recall that the notions of a family and a projective family are defined in Section 1.2.

Definition 3.1. Let C be a complex curve. We say that a point $c \in C$ is a nodal singularity of C if :

$$\widehat{\mathcal{O}}_{C,c} \cong \mathbb{C}[[x,y]]/(xy),$$

where $\widehat{\mathcal{O}}_{C,c}$ is the completion of the local ring $\mathcal{O}_{C,c}$.

Definition 3.2. *A semistable curve* is a projective curve which is connected, reduced, has only nodal singularities and such that all irreducible components isomorphic to $\mathbb{P}^1$ meet the other components in at least two points.

Definition 3.3. A morphism $f : X \to Y$ between a projective surface X which is not necessarily smooth and a smooth projective curve Y is called *a semistable map* if it is a proper, surjective morphism such that all fibers are semistable curves. The variety X is said to be semistable over Y.

Definition 3.4. Let Y be a smooth projective curve and let X be a normal projective surface. A morphism $f : X \to Y$ is called *a weakly semistable family* if it is a proper, surjective morphism such that all fibers are semistable curves.

Definition 3.5. A weakly semistable family $f : X \to Y$ is called a semistable family if X is a smooth surface.

Remark 3.6. Note that the previous definition is equivalent to the following: A projective family of curves $f : X \to Y$ over a curve Y is *a semistable family of curves* if all fibers are reduced, the singular fibers of the family are normal crossing divisors and they do not contain (-1)-curves. Let us explain the last condition:

Let $C = \sum_{j=1}^{k} C_j$ be a singular reduced fiber such that the component $C_i \cong \mathbb{P}^1$, for i fixed, and we suppose that C_i meet the other components at least at two points, i.e. $\sum_{j=1, j\neq i}^{k} C_i \cdot C_j \geq 2$. One Zariski's lemma (see §III 8.2 [8]) implies that $C_i \cdot C = 0$, and one gets:

$$C_i^2 + \sum_{j=1, j\neq i}^{k} C_i \cdot C_j = 0,$$

and hence

$$C_i^2 \neq -1.$$

Therefore, C_i is not a (-1)-curve.

Definition 3.7. Let R be a commutative Noetherian local ring. The ring R is Cohen-Macaulay if its depth is equal to the Krull dimension of R. A variety X is called Cohen-Macaulay if its local rings $\mathcal{O}_{X,x}$ at every point $x \in X$ are Cohen-Macaulay. Moreover, X is called Gorenstein if it is Cohen-Macaulay and ω_X is an invertible sheaf. A morphism $f : X \to Y$ is called Cohen-Macaulay, resp. Gorenstein, if f is flat and all the fibers are Cohen-Macaulay, resp. Gorenstein.

Definition 3.8. ([74] p.14) A normal variety X has rational singularities if it is Cohen-Macaulay and the minimal (or any) desingularization $\delta : Z \to X$ satisfies $\delta_* \omega_Z = \omega_X$. If X is a surface then rational singularities on X are called rational double points.

Remark 3.9. The name rational double points or simple surface singularities, comes from the fact that these singularities are the singularities of double coverings of a smooth surface ramified over a curve having A-D-E singularities (see §III.7 [8].) By Theorem 7.1 from §III.7 [8], the rational double points on normal surface are resolved by A-D-E curves, which are the exceptional curves whose all irreducible components are (-2)-curves and which can be seen as chains of $\mathbb{P}^1$'s .

Now, we will state several results which can be found in [60], but essentially these are the results of Mumford and Deligne.

Theorem 3.1. *(Theorem 2.2.1) Let Y be a smooth projective connected curve. Assume that X is a projective surface and $f : X \to Y$ is a surjective, proper morphism. There exists a minimal smooth surface X' birationally equivalent to X together with a flat proper morphism $f' : X' \to Y$ with the same generic fiber as f and whose fibers are normal crossing divisors on X'.*

Theorem 3.2. *(Theorem 3.2.1) Let Y be a smooth projective connected curve. Assume that X is a minimal, smooth projective surface and $f : X \to Y$ is a surjective, proper morphism whose fibers are normal crossing divisors on X. Then there exists a covering $Y' \to Y$ with Y' a smooth connected curve such that the normalization of the base change surface $X' = X \times_Y Y'$ is semistable over Y'.*

Theorem 3.3. *(Theorem 3.1.1) Let Y be a smooth projective connected curve. Let $f : X \to Y$ be a weakly semistable family. Then singularities of X can only occur at points where a fiber of f is singular. All singularities of X are rational double points. The minimal desingularization $\delta : Z \to X$ of a singular point $x \in X$ replaces x by a chain of $\mathbb{P}^1$'s, in particular the composition of f with δ is a semistable family.*

Let us put together these results in the following theorem which is known as the semistable reduction theorem, proved by Mumford in [43]. The theorem implies that if we start with any proper surjective map $f : X \to Y$, where Y is a smooth connected curve and X is a variety, there exists a semistable family after taking a base change and performing a desingularization. The similar result proved by de Jong in [38], when the relative dimension of the family is one, i.e. $\dim X - \dim Y = 1$, provides as a final result a family of nodal curves, in particular both the source and the base are smooth. Unfortunately, as it is shown in [2], such a strong result is false in the case when $\dim Y > 1$ and $\dim X - \dim Y > 1$, and the best expected result that one could have after taking well chosen base change are weakly semistable families, but in that higher dimensional case they are defined in another way, which we will not discuss here.

Here, we give the formulation of the semistable reduction theorem from [43] and then a sketch of the proof which we found in [13], but which is obviously implied by the previous several theorems due to Mumford.

Theorem 3.4. *(The semistable reduction theorem [43], p.53) Let $f : X \to Y$ be a proper surjective map of a projective surface X onto a smooth projective connected curve Y, let S be a set of points on Y such that the restriction $f : X \backslash f^{-1}(S) \to Y \backslash S$ is a smooth map. There exist:*

1. *a finite base change $\varphi : Y' \to Y$, with Y' smooth, connected totally ramified over S;*
2. *a proper birational morphism $Z \to X \times_Y Y'$, where Z is smooth and such that the induced morphism $g : Z \to Y'$ is a semistable family.*

Proof. We can blow up X at some points of $f^*(S)$ obtaining a new family $\tilde{f} : \tilde{X} \to Y$, where $\tilde{f}^*(S)$ is a normal crossing divisor, although it may not be reduced (see Theorem 3.1).

Let l be the lcm of all multiplicities of the components of $\tilde{f}^*(S)$. We make a base change of degree l, i.e. $\varphi : Y' \to Y$, totally ramified over S. Let X' be the normalization of the fiber product $\tilde{X} \times_Y Y'$. Then the composition of the normalization map and the second projection gives a map $f' : X' \to Y'$ which is a weakly semistable family (see Theorem 3.2).

Then, the singularities on X' are rational double points and the minimal resolution of singularities $Z \to X'$ introduces a chain of $\mathbb{P}^1_{\mathbb{C}}$'s which preserves the normal crossings and reducedness (see Theorem 3.3). Therefore, the induced map $Z \to Y'$ is a semistable family. □

In order to better understand the properties of weakly semistable families and semistable families let us define mild morphisms. These are the morphisms which preserves their properties under a surjective base change. The main references for this part are §2 [78] and §2 [79] .

Definition 3.10. (§2 [78]) A morphism $f : X \to Y$ between projective varieties is called mild if:

1. f is flat, Gorenstein with reduced fibers;
2. Y is smooth and X is normal with at most rational singularities;
3. given a dominant morphism $Y' \to Y$ where Y' has at most rational Gorenstein singularities, $X \times_Y Y'$ is normal with at most rational singularities.

Lemma 3.5. *(§2 [78]) Let $f : X \to Y$ be a mild morphism. If $Y' \to Y$ is a surjective morphism between projective manifolds, then $pr_2 : X \times_Y Y' \to Y'$ is a mild morphism. In particular, $X \times_Y Y'$ is normal with at most rational singularities.*

Now, we will state one result of Abramovich and Karu from §6 [2] which explains why we introduced the notion of mild morphism.

Proposition 3.6. *(p.18, [2]) Let $f : X \to Y$ be a weakly semistable family, then f is a mild morphism.*

If one considers families $f : X \to Y$, where $\dim Y > 1$ and $\dim X - \dim Y > 1$ whose fibers are reduced normal crossing divisors on X, after a base change all good properties of these families are not preserved and usually the best that one could expect are mild morphisms, §2 see [79] . Fortunately, in the cases when $\dim Y = 1$, a surjective base change $Y' \to Y$, with Y' a smooth curve, of a semistable family of curves will produce a weakly semistable family of curves, and after a desingularization one gets again a semistable family. The next theorem gathers several results of Viehweg and Zuo about the behavior of semistable families of curves after a smooth surjective base change.

Theorem 3.7. *Let $f : X \to Y$ be a semistable family of curves smooth over $U = Y \backslash S$, where S is a finite set on the curve Y. Let $\varphi : Y' \to Y$ be a finite surjective map, where Y' is a smooth curve, then:*

1. *$X' = X \times_Y Y'$ is normal with at most rational double points.*
2. *The induced map $f' : X' \to Y'$ is a weakly semistable family.*
3. *Let $\delta : Z \to X'$ be a minimal desingularization, then $g = f' \circ \delta : Z \to Y'$ is a semistable family and $g_* \omega^{\nu}_{Z/Y'} \cong f'_* \omega^{\nu}_{X'/Y'}$, for all integers $\nu \geq 1$.*

Proof.

1. This is a direct consequence of the fact that f is a mild morphism.
2. It is clear that the properness and the flatness of the map f are preserved under a base change. Also, the fibers of the map f' are semistable curves. By 1. one has that X' is normal, hence by Definition 3.4 the map $f' : X' \to Y'$ is a weak semistable family.
3. The first part of the claim is due to Theorem 3.3. The second part of the claim is a consequence of the fact that f' is a weakly semistable family, hence a mild morphism and that δ is a birational map, therefore the result follows by Corollary 2.4 vii) from §2 [78].

□

Definition 3.11. The family $g : Z \to Y'$ from the previous theorem is called the induced semistable family by the covering map $\varphi : Y' \to Y$.

Remark 3.12. Weak semistable maps are Gorenstein, since they are mild morphisms. In the notations of the previous theorem, one has the base change property:

$$\varphi'^* \omega_{X/Y} = \omega_{X'/Y'},$$

where $\varphi' : X' \to X$ is the first projection. This is a consequence of Theorem 3.6.1 from [12].

Let us finish this section by a theorem of Alan Landman which can be found in [47]. This theorem holds for semistable families of n-manifolds, but here we give the version for a family of curves:

Theorem 3.8. *([47]) Let $f : X \to Y$ be a semistable family of curves over a curve Y, with discriminant locus a finite set S. The monodromy operators of the local system $R^1 f_* \mathbb{C}_{X \backslash f^{-1} S}$ are unipotent.*

3.2 An Arakelov inequality

Beauville showed in his paper [5] that each semistable family of curves over the projective line $f : X \to \mathbb{P}^1_{\mathbb{C}}$ contains at least 4 singular fibers and he conjectured that if the genus p of the generic fiber is at least 2 the family must have at least 5 singular fibers. Later, Tan proved in [71] that for any semistable family of curves $f : X \to Y$ over a curve with fibers of genus $p \geq 2$ one has the inequality:

$$\deg f_* \omega_{X/Y} < \frac{p}{2} \deg(\Omega^1_Y(\log S)),$$

where S is the discriminant locus of the family. Combining this result with the result of Beauville he proved the conjecture. Here, we will explain some related numerical results about semistable families of curves over a curve, due to Viehweg and Zuo. The central part will be to prove an Arakelov inequality which bounds the degree of some invertible subsheaf $\mathcal{H}$ of the pushforward of the pluricanonical relative sheaf $f_* \omega^{\nu}_{X/Y}$ of a semistable family of curves $f : X \to Y$ over a curve Y. This is the inequality:

$$\deg \mathcal{H} \leq \frac{\nu}{2} \deg(\Omega^1_Y(\log S)),$$

where $\nu \geq 1$ is a positive integer and S is the discriminant locus of the family. We will call this inequality an Arakelov inequality.
The name is inspired by the paper of Arakelov [1] which treats similar problems. Peters, in his paper [58], named the class of similar inequalities Arakelov-type inequalities.

Remark 3.13. Let Y be a smooth curve and S an effective divisor on Y. Recall that whenever we write $\Omega^1_Y(\log S)$ we mean $\Omega^1_Y(\log S_{\text{red}})$, the sheaf of logarithmic differential forms with poles at most on S_{red}.

Lemma 3.9. *Let $f : X \to Y$ be a semistable family of curves over a curve Y with discriminant locus a finite set S. Let $\nu \geq 1$ and let $\mathcal{H}$ be an invertible subsheaf of $f_* \omega^{\nu}_{X/Y}$. There exists a finite cover $\varphi : Y' \to Y$, such that:*

1. *the degree of $\varphi^* \mathcal{H}$ is a multiple of ν;*
2. *let $g : Z \to Y'$ be the induced semistable family by the covering map φ, then $\varphi^* \mathcal{H}$ is an invertible subsheaf of $g_* \omega^{\nu}_{Z/Y'}$;*
3. *if $\varphi^* \mathcal{H}$ satisfies the Arakelov inequality, the same holds for $\mathcal{H}$.*

Proof. If $Y \neq \mathbb{P}^1$ then we can find an invertible non-trivial sheaf $\mathcal{F}$ on Y, exactly of order ν in $\mathrm{Pic}(Y)$, i.e. $\mathcal{F}^{\nu} = \mathcal{O}_Y$ and $\mathcal{F}^k \neq \mathcal{O}_Y$ for $0 < k < \nu$. Using the covering trick from [8] §1.17 we can construct an unramified cyclic connected covering $\varphi : Y' \to Y$ of degree ν.

Otherwise, if $Y = \mathbb{P}^1$, consider the covering $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree ν given as $z \to z^{\nu}$. The ramification points are 0 and ∞, both with multiplicity ν. The group of automorphisms of $\mathbb{P}^1$ acts transitively even on triples of points, so we can suppose that points $0, \infty$ are in the set S.

So in both cases, put $S' = \varphi^{-1} S$, and by Lemma 1.8 we have:

$$\deg(\Omega^1_{Y'}(\log S')) = \nu \deg \Omega^1_Y(\log S).$$

Since $\mathcal{H}$ is invertible and φ is a covering of compact Riemann surfaces, we have:

$$\deg(\varphi^*\mathcal{H}) = \deg\varphi \deg\mathcal{H} = \nu \deg\mathcal{H}.$$

$$\begin{tikzcd}
Z \arrow[r, bend right, "\delta"'] \arrow[d, "g"'] & X' = Y' \times_Y X \arrow[r, "\varphi'"'] \arrow[d, "f'"'] & X \arrow[d, "f"'] \\
Y' \arrow[r, "="'] & Y' \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

Let $X' = X \times_Y Y'$. We take a minimal desingularization $\delta : Z \to X'$, and we get a semistable family $g : Z \to Y'$ smooth over $Y' \backslash \varphi^{-1}S$. By Theorem 3.7 we have:

$$g_*\omega^\nu_{Z/Y'} = f'_*\omega^\nu_{X'/Y'}.$$

On the other hand, Y is a smooth curve and φ is surjective, so φ is a flat morphism, then by Proposition 9.3. from §III [32], Remark 3.12 and Theorem 3.7 we get:

$$\varphi^*\mathcal{H} \subseteq \varphi^* f_*\omega^\nu_{X/Y} = f'_*\varphi'^*\omega^\nu_{X/Y} = f'_*(\varphi'^*\omega_{X/Y})^\nu = f'_*\omega^\nu_{X'/Y'} = g_*\omega^\nu_{Z/Y'}.$$

We suppose that the Arakelov inequality holds for $\varphi^*\mathcal{H}$. Then by Lemma 1.8 one gets:

$$\nu \deg\mathcal{H} = \deg\varphi^*\mathcal{H} \leq \frac{\nu}{2}\deg(\Omega^1_{Y'}(\log S')) = \frac{\nu}{2}\nu\deg(\Omega^1_Y(\log S)),$$

and hence,

$$\deg\mathcal{H} \leq \frac{\nu}{2}\deg(\Omega^1_Y(\log S)).$$

□

The theorem of Viehweg and Zuo states that the Arakelov inequality always holds. Before we state and prove that theorem, let us give the version for a more simple case, the case of a subsheaf of the direct image of the relative canonical sheaf. That is Lemma 0.6 from [55].

Theorem 3.10. *Let $f : X \to Y$ be a semistable family of curves over a curve Y, smooth over $U = Y\backslash S$. Let $\mathcal{L}$ be an invertible subsheaf of $f_*\omega_{X/Y}$. Then:*

$$\deg\mathcal{L} \leq \frac{1}{2}\deg(\Omega^1_Y(\log S)).$$

Proof. Let $D = f^*S$. As we explained in section 2.3.2, the Higgs bundle which comes with the family $f : X \to Y$, corresponding to the local system $R^1 f_* \mathbb{C}_{X \backslash D}$ is:

$$E = E^{1,0} \oplus E^{0,1} = f_* \Omega^1_{X/Y}(\log D) \oplus R^1 f_* \mathcal{O}_X,$$

with Higgs field: $\Theta : E^{1,0} \to E^{0,1} \otimes \omega_Y(S)$. Since the family $f : X \to Y$ is semistable, by Lemma 1.9, we have $\Omega^1_{X/Y}(\log D) = \omega_{X/Y}$. Let us consider the subsheaf $L = L^{1,0} \oplus L^{0,1} \subset E$ generated by $\mathcal{L}$, i.e.

$$L^{1,0} = \mathcal{L},$$

$$L^{0,1} = \Theta(L^{1,0}) \otimes (\omega_Y(S))^{-1}.$$

By Remark 2.14 the Higgs bundle E is polystable of degree 0, therefore $\deg L \leq 0$. On the other hand, since $\mathcal{L}$ is an invertible sheaf, we have two possible cases:

1. If $\Theta(L^{1,0}) = 0$, then $L^{0,1} = 0$ and we have:

$$\deg \mathcal{L} = \deg L \leq 0 \leq \frac{1}{2} \deg(\Omega^1_Y(\log S)).$$

   Note that in the case when $Y \cong \mathbb{P}^1$ one has $\#S \geq 4$, due to [5].

2. Otherwise, $\Theta(L^{1,0}) \cong L^{1,0}$, then:

$$0 \geq \deg L = \deg L^{1,0} + \deg L^{0,1} = 2 \deg \mathcal{L} - \deg(\Omega^1_Y(\log S)),$$

   and

$$\deg \mathcal{L} \leq \frac{1}{2} \deg(\Omega^1_Y(\log S)).$$

□

The proof of the previous theorem is pretty much simplified due to the fact that the sheaf $\mathcal{L} \subseteq f_* \omega_{X/Y}$ is a subsheaf of the $(1,0)$-part of the Higgs bundle associated to the family $f : X \to Y$. For $\nu \geq 2$ and an invertible subsheaf $\mathcal{H} \subseteq f_* \omega^{\nu}_{X/Y}$ the proof that the Arakelov inequality holds is more difficult. The proof requires the application of the semistable reduction theorem and the construction of a cyclic covering of degree ν over X, in order that the sheaf $\mathcal{H}$ be a subsheaf of the $(1,0)$-part of some polystable Higgs bundle of degree 0. The following theorem is the version for the case of families of curves of Proposition 2.1 from [55] and here we fill in all details of the proof given by Viehweg and Zuo.

Theorem 3.11. *(Viehweg-Zuo theorem, [55] §2.1) Let $f : X \to Y$ be a semistable family of curves over a curve Y and smooth over $U = Y \backslash S$. Let $\nu \geq 1$ and let $\mathcal{H}$ be an invertible subsheaf of $f_* \omega^{\nu}_{X/Y}$. Then:*

$$\deg \mathcal{H} \leq \frac{\nu}{2} \deg(\Omega^1_Y(\log S)).$$

Proof. After passing to a suitable cover of Y we can suppose by Lemma 3.9 that $\deg \mathcal{H} = \nu\rho$, where ρ is a positive integer. We will now show that we can even suppose $\mathcal{H} \cong \mathcal{C}^\nu$ for a suitable line bundle $\mathcal{C}$.

Let P be a divisor on Y of degree ρ, then:

$$\deg(\mathcal{H} \otimes \mathcal{O}_Y(-\nu P)) = \nu\rho - \nu\rho = 0.$$

Then $\mathcal{H} \otimes \mathcal{O}_Y(-\nu P)$ is an element in $\mathrm{Pic}^0(Y)$.

The Jacobian $\mathrm{Pic}^0(Y)$ is a divisible group, hence there is an element $\mathcal{N}$ in this group such that:

$$\mathcal{N}^\nu \cong \mathcal{H}^{-1} \otimes \mathcal{O}_Y(\nu P).$$

We have

$$\mathcal{H} \cong \mathcal{C}^\nu,$$

where

$$\mathcal{C} = \mathcal{N}^{-1} \otimes \mathcal{O}_Y(P).$$

We are going to construct a cyclic covering of X in order that $\mathcal{C}$ belongs to the (1,0)-part of a polystable Higgs bundle of degree 0, i.e. a Higgs bundle associated to a geometric variation of Hodge structures with unipotent monodromies.
The inclusion

$$\mathcal{H} \hookrightarrow f_*\omega^\nu_{X/Y},$$

tensorised by $\mathcal{H}^{-1} = \mathcal{C}^{-\nu}$ yields an inclusion

$$\mathcal{O}_Y \hookrightarrow f_*\omega^\nu_{X/Y} \otimes \mathcal{C}^{-\nu}$$

and hence

$$f^*(\mathcal{O}_Y) = \mathcal{O}_X \hookrightarrow f^*f_*\omega^\nu_{X/Y} \otimes f^*\mathcal{C}^{-\nu}.$$

Using the canonical morphism $f^*f_*\omega^\nu_{X/Y} \to \omega^\nu_{X/Y}$, we have an injective morphism:

$$\mathcal{O}_X \to \omega^\nu_{X/Y} \otimes f^*\mathcal{C}^{-\nu} = \mathcal{L}^\nu,$$

where $\mathcal{L} = \omega_{X/Y} \otimes f^*\mathcal{C}^{-1}$.
So a non-zero constant function from $\mathcal{O}_X$ induces a non-zero section of $\mathcal{L}^\nu$. Hence, there exists a non-zero global section $\hat{\sigma}$ of $\mathcal{L}^\nu$, whose zero-divisor will be denoted by $\hat{D}$ and we fix an isomorphism:

$$\mathcal{O}_X(\hat{D}) \cong \mathcal{L}^\nu.$$

The section $\hat{\sigma} \in H^0(X, \mathcal{L}^\nu)$ defines a cyclic covering $\hat{X}$ of X, i.e. we get a cyclic covering map $\hat{\pi} : \hat{X} \to X$, see Section 1.1. Let $\hat{W} \to \hat{X}$ be a minimal resolution of singularities of $\hat{X}$ and let $\hat{\tau} : \hat{W} \to X$ be the composition of the desingularization map and the cyclic covering map. In the end, let $\hat{h} = f \circ \hat{\tau} : \hat{W} \to Y$. On the other side, we have

$$\mathcal{L} = \omega_{X/Y} \otimes f^*\mathcal{C}^{-1}$$

$$f_* f^* \mathcal{C} = f_*(\omega_{X/Y} \otimes \mathcal{L}^{-1})$$

and then by the projection formula we get:

$$\mathcal{C} \otimes f_*\mathcal{O}_X = f_*(\omega_{X/Y} \otimes \mathcal{L}^{-1})$$

$$\mathcal{C} = f_*(\omega_{X/Y} \otimes \mathcal{L}^{-1}),$$

where we use that $f_*\mathcal{O}_X = \mathcal{O}_Y$. This holds since f is proper and it has connected fibers.

Note that the map $\hat{h} : \hat{W} \to Y$ is smooth over $Y \backslash \hat{S}$, where $\hat{S}$ is a divisor on Y whose support is the union of the points of the set S and $f(\hat{\pi}(\mathrm{Sing}(\hat{X})))$, points of Y which are images of singularities of the cyclic covering $\hat{X}$ by the map $\hat{X} \to Y$. These singularities lie over the set $\mathrm{Sing}(\hat{D})$ by Proposition 1.12.

If $\hat{D} + f^{-1}\hat{S}$ was a normal crossing divisor then, by Lemma 1.16, $\omega_{X/Y} \otimes \mathcal{L}^{-1}$ would be a direct factor of $\hat{\tau}_*\Omega^1_{\hat{W}/Y}(\log \hat{h}^{-1}(\hat{S}))$, and hence we would have:

$$\mathcal{C} = f_*(\omega_{X/Y} \otimes \mathcal{L}^{-1}) \subseteq f_*\hat{\tau}_*\Omega^1_{\hat{W}/Y}(\log \hat{h}^{-1}(\hat{S})) = \hat{h}_*\Omega^1_{\hat{W}/Y}(\log \hat{h}^{-1}(\hat{S})),$$

which is the (1,0)-term of the Higgs bundle associated to the map $\hat{h} : \hat{W} \to Y$, smooth over $Y \backslash \hat{S}$.

The problem is that in general, the divisor $\hat{D} + f^{-1}\hat{S}$ is not a normal crossing divisor, which is one of the conditions to apply Lemma 1.16. Also in order to follow the proof of the previous theorem we need that the Higgs bundle associated to the map $\hat{h} : \hat{W} \to Y$ has degree 0. Since we could have non-reduced fibers after the desingularisation of the cyclic covering $\hat{W} \to \hat{X}$, the monodromies around points of $\hat{S}$ could be non-unipotent. By Theorem 2.13 the monodromies around the discriminant locus $\hat{S}$ of the map $\hat{h} : \hat{W} \to Y$ are quasi-unipotent. But by Lemma 2.14, the quasi-unipotent monodromies are not enough to conclude that the corresponding Higgs bundle is polystable of degree 0.

To get rid of these problems, we will take a ramified cover Y' of Y, and we will construct a family $h : W \to Y'$ which will have the required properties, i.e. the pullback of $\hat{D} + f^{-1}\hat{S}$ will be a normal crossings divisor and the Higgs bundle associated to the map $h : W \to Y'$ will be polystable of degree 0.

Moreover, we will prove that the pullback of $\mathcal{C}$ on Y' is an invertible subsheaf of the (1,0)-part of the Higgs bundle corresponding to the geometric variation of the map $h : W \to Y'$.

Let us explain the next diagram, which will be the first step in the construction:

$$\begin{tikzcd}
W_1 \arrow[r, bend right] \arrow[dd, bend right, "h_1"'] & \hat{W} \times_Y Y_1 \arrow[r] \arrow[d] & \hat{W} \arrow[d, "\hat{\tau}"'] \arrow[dd, bend left, "\hat{h}"] \\
Z_1 \arrow[r, bend right, "\delta_1"'] \arrow[d] & X_1 = X \times_Y Y_1 \arrow[r, "p"'] \arrow[d] & X \arrow[d, "f"'] \\
Y_1 \arrow[r, "="] & Y_1 \arrow[r, "\varphi_1"'] & Y
\end{tikzcd}$$

$\Delta = f^{-1}(S) \quad \hat{\Delta} = f^{-1}(\hat{S})$

$S \subset \hat{S}$

1. The monodromies around points $s \in \hat{S}$ of the map $\hat{h} : \hat{W} \to Y$ are quasiunipotent. The Semistable Reduction Theorem provides a covering $\varphi_1 : Y_1 \to Y$ ramified above points of $\hat{S}$ such that the normalization of $\hat{W} \times_Y Y_1$ is weakly semistable over Y_1.

Moreover, if W_1 is a minimal desingularization of the normalization of $\hat{W} \times_Y Y_1$, then the induced map $h_1 : W_1 \to Y_1$ is a semistable family and monodromies around points of $\varphi_1^{-1}\hat{S}$ are unipotent. Note that by Theorem 3.7 the fiber product $X_1 = X \times_Y Y_1$ is weakly semistable over Y_1. Singular points on $X_1 = X \times_Y Y_1$ lie on fibers over the points of $\varphi_1^{-1}S$. Since the map $f : X \to Y$ is semistable, by Theorem 3.7 these points are at most rational double points. Let $\delta_1 : Z_1 \to X_1$ be a minimal desingularization of X_1. The map $Z_1 \to Y_1$ is semistable.

Let $p : X_1 \to X$ be the first projection, then the map $\delta_1 : Z_1 \to X_1$ is an isomorphism outside of the divisor $p^*\Delta$, where $\Delta = f^{-1}(S)$. This holds since we did the desingularization at some points of $p^*\Delta$.

$$\begin{tikzcd}
Z \arrow[r, bend right] \arrow[d, "g"'] & Z_2 \times_{Y_1} Y' \arrow[r] \arrow[d] & Z_2 \arrow[r, bend right, "\delta_2"'] \arrow[d] & Z_1 \arrow[r, bend right, "\delta_1"'] \arrow[d] & X_1 = X \times_Y Y_1 \arrow[r, "p"'] \arrow[d] & X \arrow[d, "f"'] \\
Y' \arrow[r, "="'] & Y' \arrow[r, "\varphi_2"'] & Y_1 \arrow[r, "="'] & Y_1 \arrow[r, "="'] & Y_1 \arrow[r, "\varphi_1"'] & Y
\end{tikzcd}$$

2. Let us explain the construction of the diagram above:

a) Let $\hat{\Delta} = f^{-1}\hat{S}$. This divisor is a normal crossings divisor, since f is a semistable family. We recall that this divisor contains $\mathrm{Sing}(\hat{D})$, since over $\mathrm{Sing}(\hat{D})$ on the cyclic covering $\hat{W}$ lie singularities. We note that $\hat{D}$ could be a non normal crossing divisor since we don't have a control over the section $\hat{\sigma}$, whose zero divisor is the divisor $\hat{D}$. We divide $\mathrm{Sing}(\hat{D})$ in two groups:

- normal crossings of components of $\hat{D}$;
- the other points.

The divisor $\hat{D}+\hat{\Delta}$ could be a non normal crossing divisor and the points of $\mathrm{Sing}(\hat{D}+\hat{\Delta})$ are divided in two groups:

- normal crossings of $\hat{D}+\hat{\Delta}$;
- the other points, which will be referred to in the rest of the proof as "problematic points".

The problematic points of the divisor $\hat{\Delta}+\hat{D}$ are:

- the points of $\mathrm{Sing}(\hat{D})$ which are not normal crossings of components of $\hat{D}$. Above these points lie singularities on the cyclic covering, which implies that these points lie in the fibers $\hat{\Delta}=f^{-1}\hat{S}$;
- points of intersections of components of $\hat{D}$ and $\hat{\Delta}$ which are not normal crossings.

Hence, these points lie also in $\hat{\Delta}=f^{-1}\hat{S}$. A good example which describes these points is the one where 2 different components of the divisor $\hat{D}$ and a fiber over a point of S intersect at one point.
So, we get that all problematic points of the divisor $\hat{\Delta}+\hat{D}$ lie in $\hat{\Delta}$.

b) The divisor $\delta_1^*p^*(\hat{\Delta}+\hat{D})$ could be a non normal crossing divisor. By the previous observation the problematic points of the divisor $\delta_1^*p^*(\hat{\Delta}+\hat{D})$ lie at $\delta_1^*p^*(\hat{\Delta})$. More precisely, they lie in fibers over $\varphi_1^{-1}\hat{S}$.

Now, the goal is to get rid of these problematic points. In order to do that, we apply an embedded resolution at problematic points of $\delta_1^*p^*(\hat{\Delta}+\hat{D})$ on Z_1, i.e. we carry out a chain of birational maps which provides that the exceptional locus has normal crossings. Hence, we get a composition of birational maps $Z_2\to Z_1$, such that the pullback of the divisor $\hat{\Delta}+\hat{D}$ on Z_2 has normal crossings.

Note that Z_2 and Z_1 are isomorphic outside of the divisor $\delta_1^*p^*(\hat{\Delta})$, i.e. outside of the fibers over $\varphi_1^{-1}\hat{S}$. Obviously, the pullback of $\hat{D}$ to Z_2 is a normal crossings divisor with normal crossings lying in fibers over points of $\varphi_1^{-1}\hat{S}$.

Here, we have to underline that fibers over $\varphi_1^{-1}\hat{S}\backslash\varphi_1^{-1}S$ on Z_2, which are total transforms of fibers over $\varphi_1^{-1}\hat{S}\backslash\varphi_1^{-1}S$ on Z_1 after the embedded resolution $\delta_2: Z_2\to Z_1$, contain some exceptional divisors as their components. These exceptional divisors are also components of the divisor $\delta_2^*\delta_1^*p^*(\hat{D})$ and let us denote the sum of these exceptional divisors as E_{Z_2}. Hence, the divisor $\delta_2^*\delta_1^*p^*(\hat{D})$ has vertical components, at least over $\varphi_1^{-1}\hat{S}\backslash\varphi_1^{-1}S$.

c) The family $Z_2\to Y_1$ could be a non-semistable family and the multiple fibers lie over some points in $\varphi_1^{-1}\hat{S}$. But again, using the Semistable Reduction Theorem, we take a covering $\varphi_2: Y'\to Y_1$ ramified above all points of $\varphi_1^{-1}\hat{S}$ and of degree equal to the lcm of all possible multiplicities in all bad fibers. Then a minimal desingularization Z of the normalization of $Z_2\times_{Y_1}Y'$ produces a semistable family over Y', i.e. the induced map $g: Z\to Y'$ is semistable.

Moreover, the map $g : Z \to Y'$ is smooth over $Y' \backslash \varphi^{-1}(\hat{S})$, where

$$\varphi = \varphi_1 \circ \varphi_2 : Y' \to Y$$

is the composed covering map. Let us explain this:

- The singularities on X_1 are on fibers over $\varphi_1^{-1}(S)$, then after the desingularization $\delta_1 : Z_1 \to X_1$ the family $Z_1 \to Y_1$ has again discriminant locus $\varphi_1^{-1}(S)$.
- The sequence of birational maps $Z_2 \to Z_1$ will increase the discriminant locus of the family $Z_2 \to Y_1$ by transforming the smooth fibers over $\varphi_1^{-1}\hat{S} \backslash \varphi_1^{-1}S$ in the family $Z_1 \to Y_1$ into ones with normal crossings, since they get the exceptional divisors as their components after the embedded resolution. These exceptional divisors have normal crossings with the proper transforms of the fibers over $\varphi_1^{-1}\hat{S} \backslash \varphi_1^{-1}S$ in the family $Z_2 \to Y_1$. Hence, the set $\varphi^{-1}\hat{S}$ is the discriminant locus of the family $Z_2 \to Y'$.
- The set $\varphi^{-1}\hat{S}$ is also the discriminant locus for the map $Z_2 \times_{Y_1} Y' \to Y'$, and at the same time it contains singularities of $Z_2 \times_{Y_1} Y'$. These singularities are rational double points, they are resolved by chains of $\mathbb{P}^1$. The desingularization $Z \to Z_2 \times_{Y_1} Y'$ does not add new singular fibers in the family $g : Z \to Y'$.

Remark 3.14. Note that at the end of 2b) we proved that the divisor $\delta_2^*\delta_1^*p^*(\hat{D})$ has vertical components, at least over $\varphi_1^{-1}\hat{S} \backslash \varphi_1^{-1}S$, whose sum is denoted by E_{Z_2}. Also, the pullback of the divisor $\hat{D}$ to $Z_2 \times_{Y_1} Y'$, has vertical components, whose sum is the pullback of the divisor E_{Z_2} to $Z_2 \times_{Y_1} Y'$. The pullback of the divisor E_{Z_2} to $Z_2 \times_{Y_1} Y'$ has nodes as singularities and at these nodes can occur the singularities of $Z_2 \times_{Y_1} Y'$, which are rational double points. After the desingularization $Z \to Z_2 \times_{Y_1} Y'$, the pullback of the divisor $\hat{D}$ to Z contains as vertical components the sum of the pullback of the divisor E_{Z_2} to Z and the chains of $\mathbb{P}^1$'s. These vertical components lie at least over the points of $\varphi^{-1}\hat{S} \backslash \varphi^{-1}(S)$.

d) Note that the pullback of $\hat{\Delta} + \hat{D}$ on Z is a normal crossings divisor and that normal crossings of the pullback of the divisor $\hat{D}$ on Z lie in the fibers over $\varphi^{-1}\hat{S}$. Also, we have:

- Z is isomorphic to $Z_2 \times_{Y_1} Y'$ outside the pull-back of $\hat{\Delta}$;
- Z_2 and Z_1 are isomorphic outside the pull-back of $\hat{\Delta}$;
- Z_1 and X_1 are isomorphic outside the pull-back of Δ, hence also outside the pull-back of $\hat{\Delta}$;
- Note that we get a morphism $\delta : Z \to X \times_Y Y'$ from the composition of morphisms $Z \to Z_2 \times_{Y_1} Y' \to (X \times_Y Y_1) \times_{Y_1} Y'$. This map is an isomorphism outside the pullback of $\hat{\Delta}$ to Z. i.e. outside of the fibers over $\varphi^{-1}\hat{S}$.

Also, we have the maps:

- $\varphi' : X' \to X$, a finite map with ramification divisor $\hat{\Delta}$;
- $f' : X' \to Y'$, the second projection map;
- $\varphi : Y' \to Y$, a totally ramified covering with branch locus $\hat{S}$.

Let us construct the diagram:

$$
\begin{tikzcd}
W \arrow[d, "\tau"'] \arrow[dd, bend right, "h"'] & W' \arrow[r, bend right] \arrow[dd, bend right, "h'"'] & \hat{W} \times_Y Y' \arrow[r] \arrow[d] & \hat{W} \arrow[d] \arrow[dd, bend left, "\hat{h}"] & \\
Z \arrow[d, "g"'] \arrow[rr, bend right, "\delta"'] & & X' = X \times_Y Y' \arrow[r, "\varphi'"'] \arrow[d, "f'"'] & X \arrow[d, "f"'] & \Delta' = g^*T \\
Y' \arrow[r, "="] & Y' \arrow[r, "="] & Y' \arrow[r, "\varphi"'] & Y & T = \varphi^{-1}\hat{S}
\end{tikzcd}
$$

3. By construction at point 1 the map $h_1 : W_1 \to Y_1$ is semistable, so taking the bigger covering $\varphi : Y' \to Y_1 \to Y$ we preserve the semistability of the map $h' : W' \to Y'$, where W' is a minimal desingularization of the normalization of $\hat{W} \times_Y Y'$. So, the map h' has unipotent monodromies around $\varphi^{-1}\hat{S}$.

In order to simplify notations let:

$$T = \varphi^{-1}\hat{S} \text{ and } \Delta' = \delta^* \varphi'^* \hat{\Delta}.$$

In other words, we have:

$$\Delta' = g^*T.$$

Let

$$D = \operatorname{div}(\sigma') = \delta^* \varphi'^* (\hat{D})$$

be the zero divisor of the section

$$\sigma' := \delta^* \varphi'^* (\hat{\sigma}).$$

This is a global section of the sheaf

$$\mathcal{M} := \delta^* \varphi'^* \mathcal{L},$$

and we have

$$\mathcal{M}^\nu \cong \mathcal{O}_Z(D).$$

So, we can construct the ν-cyclic covering $\pi : W_2 \to Z$ obtained by taking a ν-th root out of D. Since D is a normal crossing divisor, singularities on W_2 are rational double points and they lie over normal crossings of the divisor D, see Lemma 1.6 and Lemma 1.7. These singularities are resolved by chains of $\mathbb{P}^1$'s and the resolution is minimal. Let us denote this desingularization by $\mu : W \to W_2$. Since the normal crossings of D lie in $\Delta' = \delta^* \varphi'^* \hat{\Delta}$, the images of these normal crossings by the map $g : Z \to Y'$ on Y' are in the set T. Hence, the singularities of the cyclic covering lie over T.

We denote by $\tau = \pi \circ \mu : W \to Z$ the map which is the composition of the desingularization and the cyclic covering map. Hence, we get an induced map $h = g \circ \tau : W \to Y'$. By construction, the map h is smooth over $Y' \backslash T$. This holds since the support of the divisor T is the union of the set of images on Y' of singularities of the cyclic covering by the map g and the discriminant locus of the map $g : Z \to Y'$.

We end this part by showing that the map $h : W \to Y'$ has unipotent monodromies around T. In order to study monodromies of h, we study monodromies of h'. We will see that W and W' are isomorphic away from fibers over T. This will be sufficient because the loops along which we calculate monodromies are in the complement of pullbacks of T on W and W'. Since $h' : W' \to Y'$ has unipotent monodromies around T, we get the same for monodromies of the map $h : W \to Y'$ around T. Let us prove that W and W' are isomorphic outside the fibers over T.

The surface W is a desingularization of the normalization of Z in the function field of $\hat{W} \times_Y Y'$, i.e. W is a desingularization of the normalization of $Z \times_{X'} (\hat{W} \times_Y Y')$. By point 2. of the construction, Z and X' are isomorphic outside the divisor $\Delta' = g^*T = \delta^* \varphi'^* \hat{\Delta}$, so we get that W is a desingularization of the normalization of $(\hat{W} \times_Y Y')$ outside the fibers over T. Recall that W' is a desingularization of the normalization of $(\hat{W} \times_Y Y')$. In the end, we have an isomorphism of W and W' outside the fibers over T.

We summarize what we have seen so far in the following claim:

Claim 3.12. *In the previous notations, the map $g : Z \to Y'$ is smooth over $Y' \backslash T$ and the map $h : W \to Y'$ is smooth over $Y' \backslash T$. The covering map $\varphi : Y' \to Y$ is ramified over the points of the set $\hat{S}$. The local monodromies of the local system $R^1 h_* \mathbb{C}_{W \backslash h^{-1}(T)}$ are unipotent. The divisor $(\Delta' + D)_{red}$ is a normal crossings divisor. The divisor D contains vertical components at least over $T \backslash \varphi^{-1} S$.*

Proof. Indeed, since the smooth varieties W and W' are isomorphic outside of divisors which are the pull-backs of the divisor T by maps h and h', and since local monodromies of the local system $R^1 h'_* \mathbb{C}_{W' \backslash h'^{-1}(T)}$ are unipotent by construction, the local monodromies of the local system $R^1 h_* \mathbb{C}_{W \backslash h^{-1}(T)}$ are unipotent as well. Also, the pullback of the divisor $\hat{\Delta} + \hat{D}$ on Z is the divisor $\Delta' + D$, hence it has normal crossings by construction. The rest follows by points 2b), 2c) and Remark 3.14 from the construction. $\square$

We divide the remainder of the proof of the theorem in several steps. In the first few parts we will give some technical details which are mostly consequences of the previous construction. Then, we will show that the invertible sheaf $\varphi^*\mathcal{C}$ can be seen as the (1,0)-part of a polystable Higgs bundle of degree 0, which will be enough to show the inequality.

(A) Since $f : X \to Y$ is semistable, by Theorem 3.7, Definition 3.8 and the projection formula, we have:

$$\delta_*(\omega_{Z/X'}) = \delta_*(\omega_Z \otimes \delta^*\omega_{X'}^{-1}) = \mathcal{O}_{X'}.$$

We have $f'_*(\mathcal{O}_{X'}) = \mathcal{O}_{Y'}$. This holds since f' is proper and it has connected fibers. These facts and Remark 3.12 yield:

$$\begin{aligned} g_*(\omega_{Z/Y'} \otimes \delta^*\varphi'^*\omega_{X/Y}^{-1}) &= g_*(\omega_Z \otimes g^*\omega_{Y'}^{-1} \otimes \delta^*\omega_{X'/Y'}^{-1}) \\ &= g_*(\omega_Z \otimes g^*\omega_{Y'}^{-1} \otimes \delta^*(\omega_{X'}^{-1} \otimes f'^*\omega_{Y'})) \\ &= g_*(\omega_Z \otimes g^*\omega_{Y'}^{-1} \otimes \delta^*\omega_{X'}^{-1} \otimes g^*\omega_{Y'}) \\ &= g_*(\omega_{Z/X'}) = f'_*\delta_*(\omega_{Z/X'}) = f'_*(\mathcal{O}_{X'}) = \mathcal{O}_{Y'}. \end{aligned}$$

(B) Let us recall that $S \subset \hat{S}$, $\Delta = f^{-1}S$, $\hat{\Delta} = f^{-1}\hat{S}$, $T = \varphi^{-1}(\hat{S})$, and $\Delta' = g^{-1}T$. Note that the map $\varphi' : X' \to X$ is a finite map with ramification divisor $\hat{\Delta}$, the map δ is a desingularization and by Theorem 1.11 we have:

$$\delta^*\varphi'^*\left(\Omega_X^1(\log\hat{\Delta})\right) \subset \Omega_Z^1\left(\log\Delta'\right). \tag{3.1}$$

Since $\varphi : Y' \to Y$ is ramified over $\hat{S}$ then by Lemma 1.8 we have:

$$\delta^*\varphi'^*f^*\Omega_Y^1(\log\hat{S}) = g^*\varphi^*\Omega_Y^1(\log\hat{S}) = g^*\Omega_{Y'}^1(\log T). \tag{3.2}$$

Therefore, for the quotients we get:

$$\delta^*\varphi'^*\Omega_{X/Y}^1(\log\hat{\Delta}) \subset \Omega_{Z/Y'}^1\left(\log\Delta'\right). \tag{3.3}$$

The map $f : X \to Y$ is smooth over $Y\backslash S$, then by Remark 1.26 we have:

$$\omega_{X/Y} = \Omega_{X/Y}^1(\log\Delta) = \Omega_{X/Y}^1(\log\hat{\Delta}),$$

which implies:

$$\delta^*\varphi'^*(\Omega_{X/Y}^1(\log\Delta)) \subset \Omega_{Z/Y'}^1\left(\log\Delta'\right). \tag{3.4}$$

(C) Tensorising with $\omega_{X/Y}^{-1}$ the sequence :

$$0 \to f^*\Omega_Y^1(\log S) \to \Omega_X^1(\log\Delta) \to \Omega_{X/Y}^1(\log\Delta) \to 0,$$

one gets the long exact sequence obtained by taking f_* with the edge morphism

$$\Theta : \mathcal{O}_Y \to R^1f_*(\omega_{X/Y}^{-1}) \otimes \omega_Y(S).$$

This map Θ is injective or zero. Therefore, when it is not zero we have an inclusion of sheaves $(\omega_Y(S))^{-1} \subset R^1f_*(\omega_{X/Y}^{-1})$.

(D) Now let us come back to the map $h : W \to Y'$ and the Higgs bundle associated to this family. The main technical tool in this part will be Lemma 1.16 from Section 1.3. It holds by Claim 3.12 that the family $h : W \to Y'$ satisfies all the conditions described at the beginning of Section 1.3 which are needed in order to apply Lemma 1.16. By this we mean:

- the family $g : Z \to Y'$ is semistable;
- $\Delta' + D$ is a normal crossings divisor;
- one has that $\pi^*((\Delta' + D) \setminus \mathrm{Sing}(D))$ is a normal crossing divisor, see Remark 1.14. Since $\mu : W \to W_2$ is the minimal desingularization which resolves the rational double points on the cyclic covering one gets that $\mu^*(\pi^*((\Delta' + D) \setminus \mathrm{Sing}(D))) + \mu^*(\pi^{-1}(\mathrm{Sing}(D)))$ is a normal crossing divisor, see Remark 3.9. Therefore, $\tau^*(\Delta' + D)$ is a normal crossing divisor.

As we already saw, by 2(b), the divisor D may have vertical components. Let us write $D = D_{ver} + D_{hor}$, where D_{ver} is the sum of vertical components and D_{hor} is the sum of horizontal components, see Definition 1.19. The support of the divisor D_{ver} is in the support of the divisor Δ'. Let Γ' be the reduced divisor whose support is the union of the components of D_{hor} whose multiplicities are not divisible by ν. Then by Lemma 1.5 the divisor $\Gamma' + \Delta'$ contains the branch locus of the cyclic covering. We will use the notations:

- $D_{hor} = \sum \alpha_j D_j$;
- $\Gamma'_i = \sum\limits_{\frac{\alpha_j i}{\nu} \notin \mathbb{Z}} D_j$, where the support of D_j is in the support of D_{hor};
- $\mathcal{M}^{(-i)} = \mathcal{M}^{-i} \otimes \mathcal{O}_Z\left(\left[\dfrac{iD}{\nu}\right]\right)$.

Since local monodromies around points of T of the local system $R^1 h_* \mathbb{C}_{W \setminus h^{-1}(T)}$ on Y' are unipotent, by Lemma 2.14 one obtains a polystable Higgs bundle of degree 0 :

$$\bigoplus_{p+q=1} R^q h_* \Omega^p_{W/Y'}(\log \tau^* \Delta').$$

By Lemma 1.16, one has the decomposition:

$$\begin{aligned}
&\bigoplus_{p+q=1} R^q h_* \Omega^p_{W/Y'}(\log \tau^*\Delta') \\
&= \bigoplus_{p+q=1} R^q g_* \left(\tau_* \Omega^p_{W/Y'}(\log \tau^*\Delta') \right) \\
&= \bigoplus_{p+q=1} R^q g_* \left(\Omega^p_{Z/Y'}(\log \Delta') \oplus \bigoplus_{i=1}^{\nu-1} \Omega^p_{Z/Y'}(\log(\Delta' + \Gamma'_i)) \otimes \mathcal{M}^{(-i)} \right) \\
&= \bigoplus_{p+q=1} R^q g_* (\Omega^p_{Z/Y'}(\log \Delta')) \oplus \bigoplus_{i=1}^{\nu-1} \bigoplus_{p+q=1} R^q g_* \left(\Omega^p_{Z/Y'}(\log(\Delta' + \Gamma'_i)) \otimes \mathcal{M}^{(-i)} \right).
\end{aligned} \tag{3.5}$$

Let us consider the case $i = 1$. Then we have $\Gamma'_1 = \Gamma'$. In that case on Y' we get the Higgs bundle

$$G = (G^{1,0} \oplus G^{0,1}, \Theta_G),$$

defined as:

$$G^{1,0} = g_*(\Omega^1_{Z/Y'}(\log(\Delta' + \Gamma')) \otimes \mathcal{M}^{(-1)}),$$
$$G^{0,1} = R^1 g_*(\mathcal{M}^{(-1)}).$$

The Higgs field $\Theta_G : G^{1,0} \to G^{0,1} \otimes \Omega^1_{Y'}(\log T)$ is induced by the edge morphism of the long exact sequence obtained by taking g_* of the sequence:

$$0 \to g^*\Omega^1_{Y'}(\log T) \to \Omega^1_Z(\log(\Delta' + \Gamma')) \to \Omega^1_{Z/Y'}(\log(\Delta' + \Gamma')) \to 0$$

tensorised with $\mathcal{M}^{(-1)}$, i.e. of the sequence:

$$\begin{aligned}
0 \to g^*\Omega^1_{Y'}(\log T) \otimes \mathcal{M}^{(-1)} &\to \Omega^1_Z(\log(\Delta' + \Gamma')) \otimes \mathcal{M}^{(-1)} \\
&\to \Omega^1_{Z/Y'}(\log(\Delta' + \Gamma')) \otimes \mathcal{M}^{(-1)} \to 0.
\end{aligned} \tag{3.6}$$

Let us prove that G is polystable and of degree zero:
In the book [59] §2, it can be found that the $\mathbb{Z}/\nu$-action on W_2 gives the decomposition:

$$\pi_* \mathbb{C}_{W_2 \setminus \pi^{-1}(\Delta')} = \tau_* \mathbb{C}_{W \setminus \tau^{-1}(\Delta')} = \bigoplus_{j=0}^{\nu-1} V_j,$$

where the V_j are rank 1 local systems on $Z \setminus g^{-1}(T)$ given by eigenspaces of the $\mathbb{Z}/\nu$-action. Let $\mathbb{W} = R^1 h_* \mathbb{C}_{W \setminus \tau^{-1}(\Delta')}$, then we have a decomposition of $\mathbb{W}$ induced by this action on W_2 :

$$\begin{aligned}\mathbb{W} &= R^1 h_* \mathbb{C}_{W\setminus\tau^{-1}(\Delta')} = R^1 g_* \left(\tau_* \mathbb{C}_{W\setminus\tau^{-1}(\Delta')}\right) \\ &= R^1 g_* (\bigoplus_{j=0}^{\nu-1} V_j) = \bigoplus_{j=0}^{\nu-1} R^1 g_* V_j = \bigoplus_{j=0}^{\nu-1} \mathbb{W}_j\end{aligned}$$

So the $\mathbb{Z}/\nu$-action induces a decomposition of $\mathbb{W}$ in a direct sum of subvariations of Hodge structures on $Y'\setminus T$ with unipotent monodromies around points of T. The Higgs bundle G corresponds to one of them. By Theorem 2.11 we get that the Higgs bundle G is polystable of degree 0.

(E) Recall the exact sequence:

$$0 \to f^*\Omega^1_Y(\log S) \to \Omega^1_X(\log \Delta) \to \Omega^1_{X/Y}(\log \Delta) \to 0,$$

whose pullback by the map $\varphi' \circ \delta$ on Z tensorised with $\mathcal{M}^{-1} = \delta^* \varphi'^* \left(\omega^{-1}_{X/Y} \otimes f^*\mathcal{C}\right)$ gives the sequence:

$$\begin{aligned}0 \to \delta^*\varphi'^*(f^*\omega_Y(S) \otimes \omega^{-1}_{X/Y} \otimes f^*\mathcal{C}) \to \delta^*\varphi'^*(\Omega^1_X(\log \Delta) \otimes \omega^{-1}_{X/Y} \otimes f^*\mathcal{C}) \\ \to \delta^*\varphi'^*(f^*\mathcal{C}) \to 0.\end{aligned} \tag{3.7}$$

The holds since $\omega_{X/Y} = \Omega^1_{X/Y}(\log \Delta)$. The long exact sequence associated to exact sequence (3.7) obtained by taking g_* induces the Higgs bundle J on Y' defined as:

$$J^{1,0} = g_* \left(\delta^*\varphi'^*(f^*(\mathcal{C}))\right) = \varphi^*(\mathcal{C}),$$

$$\begin{aligned}J^{0,1} = R^1 g_* \left(\delta^*\varphi'^*(\omega^{-1}_{X/Y} \otimes f^*(\mathcal{C}))\right) &= R^1 g_* \left(\delta^*\varphi'^*(\omega^{-1}_{X/Y})\right) \otimes \varphi^*(\mathcal{C}) \\ &= \varphi^* R^1 f_*(\omega^{-1}_{X/Y}) \otimes \varphi^*(\mathcal{C}).\end{aligned}$$

The last equality holds by Proposition 9.3 Chapter III from [32]. The Higgs field is induced by the edge morphism of the long exact sequence:

$$\Theta_J : J^{1,0} \to J^{0,1} \otimes \varphi^*(\omega_Y(S)) \subset J^{0,1} \otimes \omega_{Y'}(\varphi^* S).$$

(F) In this point we will prove that there is a morphism between Higgs bundles J and G. By (3.1), (3.2), (3.4) the sequence:

$$0 \to \delta^*\varphi'^*(f^*\Omega^1_Y(\log S)) \to \delta^*\varphi'^*(\Omega^1_X(\log \Delta)) \to \delta^*\varphi'^*(\Omega^1_{X/Y}(\log \Delta)) \to 0$$

is a subsequence of

$$0 \to g^*\Omega^1_{Y'}(\log T) \to \Omega^1_Z(\log \Delta') \to \Omega^1_{Z/Y'}(\log \Delta') \to 0 \tag{3.8}$$

and so a subsequence of

$$0 \to g^*\Omega^1_{Y'}(\log T) \to \Omega^1_Z(\log(\Delta' + \Gamma')) \to \Omega^1_{Z/Y'}(\log(\Delta' + \Gamma')) \to 0.$$

This holds since

$$\Omega^1_Z(\log(\Delta' + \Gamma')) \supset \Omega^1_Z(\log \Delta'),$$

and so

$$\Omega^1_{Z/Y'}(\log(\Delta' + \Gamma')) \supset \Omega^1_{Z/Y'}(\log \Delta').$$

This yields the commutative diagram:

$$\begin{tikzcd}
\delta^*{\varphi'}^*(f^*\Omega^1_Y(\log S)) \arrow[r] \arrow[d, hook, "\cap"] & \delta^*{\varphi'}^*(\Omega^1_X(\log \Delta)) \arrow[r] \arrow[d, hook, "\cap"] & \delta^*{\varphi'}^*(\Omega^1_{X/Y}(\log \Delta)) \arrow[r] \arrow[d, hook, "\cap"] & 0 \\
g^*\Omega^1_{Y'}(\log T) \arrow[r] & \Omega^1_Z(\log(\Delta' + \Gamma')) \arrow[r] & \Omega^1_{Z/Y'}(\log(\Delta' + \Gamma')) \arrow[r] & 0
\end{tikzcd}$$

The vertical maps in the diagram are inclusion maps. We construct a new commutative diagram by tensorising the first sequence with

$$\mathcal{M}^{-1} = \delta^*{\varphi'}^*\left(\omega^{-1}_{X/Y} \otimes f^*\mathcal{C}\right),$$

and the second with

$$\mathcal{M}^{(-1)} = \delta^*{\varphi'}^*\left(\omega^{-1}_{X/Y} \otimes f^*\mathcal{C}\right) \otimes \mathcal{O}_Z\left(\left[\frac{D}{\nu}\right]\right).$$

The rows in that new diagram are the sequences (3.7) and (3.6). The vertical maps will still be inclusion maps since $\mathcal{M}^{-1} \subset \mathcal{M}^{(-1)}$. Moreover, the long exact sequences obtained by taking g_* of rows of that diagram give the new commutative "long" diagram. The squares of the long diagram are commutative. The third square in that diagram is:

$$\begin{tikzcd}
J^{1,0} \arrow[r, "\Theta_J"] \arrow[d, "\epsilon^{1,0}"] & J^{0,1} \otimes \varphi^*(\omega_Y(S)) \arrow[d, "\epsilon^{0,1} \otimes i"] \\
G^{1,0} \arrow[r, "\Theta_G"] & G^{0,1} \otimes \omega_{Y'}(T)
\end{tikzcd}$$

This holds by construction of the Higgs bundles G and J done in points (D) and (E). Since $\Theta_J : J^{1,0} \to J^{0,1} \otimes \varphi^*(\omega_Y(S)) \subset J^{0,1} \otimes \omega_{Y'}(\varphi^* S)$ (see Lemma 1.8) we get the following commutative diagram:

$$\begin{tikzcd} J^{1,0} \arrow[r, "\Theta_J"] \arrow[d, "\epsilon^{1,0}"] & J^{0,1} \otimes \omega_{Y'}(\varphi^* S) \arrow[d, "\epsilon^{0,1} \otimes i"] \\ G^{1,0} \arrow[r, "\Theta_G"] & G^{0,1} \otimes \omega_{Y'}(T) \end{tikzcd}$$

The commutativity of this diagram implies the existence of a morphism between the Higgs bundles $\epsilon : J \to G$, see Definition 2.19. Note that i is an inclusion map.

(G) We prove that the Higgs subbundle A of J generated by the sheaf $\varphi^*\mathcal{C}$ is a Higgs subbundle of the polystable Higgs bundle G of degree 0. By definition,

$$A^{1,0} = \varphi^*\mathcal{C} = J^{1,0},$$

and

$$A^{0,1} = \mathrm{Im}(\Theta_{J|A^{1,0}}) \otimes (\varphi^*(\omega_Y(S)))^{-1} \subset J^{0,1}$$

Since $A^{1,0}$ is a line bundle, Θ_J is injective or the zero map on sheaves. So we will distinguish two cases:

($\mathbf{G}_1$) In the case when Θ_J is not the zero map on $A^{1,0}$ we have:

$$A^{0,1} = \varphi^*\mathcal{C} \otimes (\varphi^*(\omega_Y(S)))^{-1} \subset J^{0,1}.$$

The Higgs field of the Higgs bundle A is given as:

$$\Theta_A : A^{1,0} \cong A^{0,1} \otimes \varphi^*(\omega_Y(S)) \subset A^{0,1} \otimes \omega_{Y'}(\varphi^* S).$$

On the other hand, using the projection formula and point A), we have :

$$\begin{aligned} G^{1,0} &= g_* \left(\Omega^1_{Z/Y'}(\log(\Delta' + \Gamma')) \otimes \mathcal{M}^{(-1)}\right) \\ &\supset g_* \left(\Omega^1_{Z/Y'}(\log \Delta') \otimes \mathcal{M}^{-1}\right) \\ &= g_*(\omega_{Z/Y'} \otimes \mathcal{M}^{-1}) \\ &= g_* \left(\omega_{Z/Y'} \otimes \delta^* {\varphi'}^* \omega_{X/Y}^{-1} \otimes \delta^* {\varphi'}^* f^*(\mathcal{C})\right) \\ &= g_* \left(\omega_{Z/Y'} \otimes \delta^* {\varphi'}^* \omega_{X/Y}^{-1}\right) \otimes \varphi^*(\mathcal{C}) \\ &= \mathcal{O}_{Y'} \otimes \varphi^*(\mathcal{C}) \\ &= \varphi^*(\mathcal{C}). \end{aligned}$$

Hence, $A^{1,0}$ is a subbundle of $G^{1,0}$. Also, we have an inclusion map $A^{0,1} \hookrightarrow J^{0,1}$ and a morphism $\epsilon^{0,1} : J^{0,1} \to G^{0,1}$, whose composition gives a map $A^{0,1} \to G^{0,1}$. We have the commutative diagram:

$$\begin{tikzcd}
J^{1,0} = A^{1,0} \arrow[r, "\Theta_A"] \arrow[rr, bend left, "\Theta_J"] \arrow[d, "\epsilon^{1,0}"] & A^{0,1} \otimes \varphi^*(\omega_Y(S)) \arrow[r] \arrow[d] & J^{0,1} \otimes \varphi^*(\omega_Y(S)) \arrow[dl, "\epsilon^{0,1} \otimes i"] \\
G^{1,0} \arrow[r, "\Theta_G"] & G^{0,1} \otimes \omega_{Y'}(T) &
\end{tikzcd}$$

Also, we note :

$$\Theta_A : A^{1,0} \cong A^{0,1} \otimes \varphi^*(\omega_Y(S)) \subset A^{0,1} \otimes \omega_{Y'}(\varphi^* S) \subset A^{0,1} \otimes \omega_{Y'}(T).$$

Therefore, the Higgs bundle A is a Higgs subbundle of G. Moreover, its Higgs field has no poles over $T \backslash \varphi^{-1}(S)$.
In the end, from $\deg \varphi^*(\mathcal{C}) = \frac{1}{\nu} \deg \varphi^* \mathcal{H}$ we get:

$$\begin{aligned}
\deg A = \deg A^{1,0} + \deg A^{0,1} &= \deg \varphi^*(\mathcal{C}) + \deg \varphi^*(\mathcal{C} \otimes \omega_Y(S)^{-1}) \\
&= \deg \varphi^*(\mathcal{C}) + \deg \varphi^*(\mathcal{C}) - \deg \varphi^*(\omega_Y(S)) \\
&= \frac{2}{\nu} \deg \varphi^* \mathcal{H} - \deg \varphi^*(\omega_Y(S)) \\
&= \deg \varphi \left(\frac{2}{\nu} \deg \mathcal{H} - \deg(\omega_Y(S) \right).
\end{aligned}$$

The Higgs bundle G is polystable of degree zero, hence Higgs subbundles of G must have degree less than or equal to 0. Thus, for the Higgs subbundle A of G we get $\deg A \leq 0$. Therefore,

$$\frac{2}{\nu} \deg \mathcal{H} - \deg(\omega_Y(S)) \leq 0,$$

$$\deg \mathcal{H} \leq \frac{\nu}{2} \deg(\omega_Y(S)).$$

($\mathbf{G}_2$) In the case when Θ_J is the zero map on $A^{1,0}$, by the construction of the Higgs subbundle A one has $A^{0,1} = 0$ and $A = A^{1,0}$ is a Higgs subbundle of G. Since G is polystable of degree 0 we have

$$\deg A = \deg A^{1,0} = \deg \varphi^* \mathcal{C} \leq 0.$$

Therefore,

$$\deg \mathcal{H} \leq 0,$$

and thus again $\deg \mathcal{H} \leq \frac{\nu}{2} \deg(\omega_Y(S))$.

□

Remark 3.15. The Kodaira-Spencer map of the family $f : X \to Y$ is

$$\Theta_{KS} : \mathcal{O}_Y \to R^1 f_*(\omega_{X/Y}^{-1}) \otimes \omega_Y(S),$$

hence

$$\Theta_{KS} : (\omega_Y(S))^{-1} \to R^1 f_*(\omega_{X/Y}^{-1}),$$

this map is injective or zero on sheaves, so if it is injective we have:

$$(\omega_Y(S))^{-1} \subset R^1 f_*(\omega_{X/Y}^{-1}).$$

Now, tensorising with $\omega_{X/Y}^{-1} \otimes f^*\mathcal{C}$ the sequence:

$$0 \to f^*\Omega^1_Y(\log S) \to \Omega^1_X(\log \Delta) \to \Omega^1_{X/Y}(\log \Delta) \to 0,$$

and taking the long exact sequence, we get the edge morphism

$$\Theta_0 : \mathcal{C} \to \mathcal{C} \otimes R^1 f_*(\omega_{X/Y}^{-1}) \otimes \omega_Y(S).$$

This edge morphism is obtained as the tensor product with the Kodaira-Spencer map, i.e.

$$\Theta_0 : \mathcal{C} = \mathcal{C} \otimes \mathcal{O}_Y \xrightarrow{\mathrm{id}_{\mathcal{C}} \otimes \Theta_{KS}} \mathcal{C} \otimes R^1 f_*(\omega_{X/Y}^{-1}) \otimes \omega_Y(S).$$

Therefore, the map Θ_0 is:

$$\Theta_0 = \mathrm{id}_{\mathcal{C}} \otimes \Theta_{KS}$$

and we have the Higgs bundle H_0 on Y defined by:

$$H_0^{1,0} = \mathcal{C},$$

$$H_0^{0,1} = \mathcal{C} \otimes \omega_Y(S)^{-1},$$

with Higgs field

$$\Theta_0 : H_0^{1,0} \to H_0^{0,1} \otimes \omega_Y(S).$$

We note that the Higgs subbundle A of G constructed in the previous proof at point (G) is the pullback of the Higgs bundle H_0 by the map φ, with Higgs field the pullback of the map Θ_0 by the map φ. Hence, the pullback of the Kodaira-Spencer map of the family $f : X \to Y$ can be seen as a part of the Higgs field Θ_G of the Higgs bundle G associated to the geometric subvariation of Hodge structures of the family $h : W \to Y'$.

3.3 The maximal case

In this section we will investigate the case when the Arakelov bound is reached. This is what we call the maximal case.

Let $f : X \to Y$ be a semi-stable family of curves with discriminant locus S. We assume that $\chi(Y\backslash S) < 0$. Let $\nu \geq 1$ be an integer and $\mathcal{H}$ be an invertible subsheaf of $f_*\omega^{\nu}_{X/Y}$ which satisfies the Arakelov equality:

$$\deg \mathcal{H} = \frac{\nu}{2} \deg(\omega_Y(S)). \tag{3.9}$$

Lemma 3.13. *In the previous notations, if $\nu = 1$ then $Y\backslash S$ is a Teichmüller curve for the family $f : X \to Y$.*

Proof. Let $\Delta = f^{-1}S$. Since the family $f : X \to Y$ is semistable, one has $\omega_{X/Y} = \Omega^1_{X/Y}(\log \Delta)$. We denote by F the polystable Higgs bundle of degree 0 induced by the geometric variation of the semistable family $f : X \to Y$,

$$F = F^{1,0} \oplus F^{0,1} = f_*\omega_{X/Y} \oplus R^1 f_*\mathcal{O}_X,$$

with Higgs field :

$$\Theta : F^{1,0} \to F^{0,1} \otimes \omega_Y(S).$$

Let H_0 be the Higgs subbundle of F generated by $\mathcal{H}$, i.e.

$$H_0^{1,0} = \mathcal{H} \subset F^{1,0},$$

$$H_0^{0,1} = \Theta(\mathcal{H}) \otimes (\omega_Y(S))^{-1} \subset F^{0,1}.$$

Since $\mathcal{H}$ is an invertible sheaf, Θ restricted to $\mathcal{H}$ is injective or the zero map. It is not the zero map since H_0 is a Higgs subbundle of the polystable Higgs bundle F of degree 0, hence it has the degree less or equal to 0. The map Θ is an injective map and

$$H_0^{0,1} \cong \mathcal{H} \otimes (\omega_Y(S))^{-1}.$$

The Higgs field of H_0 is

$$\Theta_0 = \Theta_{|\mathcal{H}}.$$

The equality (3.9) implies:

$$\deg H_0 = \deg \mathcal{H} + \deg(\mathcal{H} \otimes (\omega_Y(S))^{-1}) = 2\deg \mathcal{H} - \deg(\omega_Y(S)) = 0.$$

Therefore, H_0 is a direct factor of the polystable Higgs bundle F. In fact, we have a decomposition:

$$F = H_0 \oplus N,$$

where N is a Higgs bundle of degree 0. This is equivalent to a decomposition of the local system

$$R^1 f_*\mathbb{C}_{X\backslash f^{-1}S} = \mathbb{H}_0 \oplus \mathbb{N},$$

by Simpson's correspondence theorem, where $\mathbb{H}_0$ is a local system which corresponds to H_0 and $\mathbb{N}$ corresponds to N. Then the Higgs bundle H_0 corresponds to a subvariation of the geometric variation $R^1 f_* \mathbb{C}_{X \backslash \Delta}$. Moreover, the fact that

$$\deg H_0^{1,0} = \frac{1}{2} \deg(\omega_Y(S)),$$

implies that the Higgs field Θ_0 is an isomorphism by Theorem 2.19. We get that $\mathbb{H}_0$ corresponds to a subvariation of $R^1 f_* \mathbb{C}_{X \backslash \Delta}$ with Higgs field an isomorphism, hence $Y \backslash S$ is a Teichmüller curve for the family $f : X \to Y$ by Theorem 2.22. $\square$

Now, we will investigate the maximal case when $\nu \geq 2$. As we have already seen in the proof of Theorem 3.11, up to Lemma 3.9, we can suppose that:

$$\mathcal{H} = \mathcal{C}^\nu,$$

where $\mathcal{C}$ is a line bundle on Y. In general, there is no reason why $\mathcal{C} \subseteq f_* \omega_{X/Y}$, so we will repeat the construction from the proof of Theorem 3.11.

$$\begin{tikzcd}
W \arrow[d, "\tau"] \arrow[dd, bend right, "h"'] & W' \arrow[dd, bend right, "h'"'] \arrow[r, bend right] & \hat{W} \times_Y Y' \arrow[r] \arrow[d] & \hat{W} \arrow[d] \arrow[dd, bend left, "\hat{h}"] & \\
Z \arrow[d, "g"] \arrow[rr, bend right, "\delta"] & & X' = X \times_Y Y' \arrow[r, "\varphi'"'] \arrow[d, "f'"] & X \arrow[d, "f"] & \Delta' = g^*T \\
Y' \arrow[r, "="] & Y' \arrow[r, "="] & Y' \arrow[r, "\varphi"'] & Y & T = \varphi^{-1}\hat{S}
\end{tikzcd}$$

Let us recall a few details about this construction. We have $\hat{W}$ the desingularization of the cyclic covering on X obtained by $\mathcal{L}^\nu = \mathcal{O}_X(\hat{D})$, where $\mathcal{L} = \omega_{X/Y} \otimes \mathcal{C}^{-1}$. The pullback of $\mathcal{L}$ to Z gives rise to a cyclic covering defined by $\mathcal{M}^\nu = \mathcal{O}_Z(D)$, where $\mathcal{M} = \delta^* \varphi^{*\prime}(\mathcal{L})$, and we denote by W the desingularization of that covering. The induced map $h : W \to Y'$ is smooth over $Y' \backslash T$, where $T = \varphi^{-1}(\hat{S}) = \varphi^{-1} S \cup \Sigma$, and Σ is a set of points on Y' which are images of the singularities of the cyclic covering on Y' by the corresponding map. By Claim 3.12 the monodromies around the points of T of the map h are unipotent. This implies the existence of a canonical extension, namely the Deligne extension, of the flat bundle $\mathcal{O}_{Y' \backslash T} \otimes R^1 h_* \mathbb{C}_{W \backslash h^{-1}(T)}$, which gives rise to a polystable Higgs bundle of degree 0:

$$E_W = \bigoplus_{p+q=1} R^q h_* \Omega^p_{W/Y'}(\log h^{-1}(T)).$$

Following 3.5 from the part (D) of the proof of Theorem 3.11, we have the decomposition of the Higgs bundle E_W into a direct sum of polystable Higgs bundles of degree 0, induced by the $\mathbb{Z}/\nu$-action on the cyclic covering:

$$E_W = \bigoplus_{i=0}^{\nu-1} G_i,$$

$$E_W = E_Z \oplus G \oplus \bigoplus_{i=2}^{\nu-1} G_i, \tag{3.10}$$

where $G_0 = E_Z$ is the Higgs bundle associated to the semistable family $g : Z \to Y'$ which is smooth over $Y' \backslash T$ and $G_1 = G$ is the Higgs bundle whose description is also given in the point (D).
Let us recall that the Higgs bundle A, which is a Higgs subbundle of G, is defined by:

$$A^{1,0} = \varphi^* \mathcal{C},$$
$$A^{0,1} = \varphi^* \mathcal{C} \otimes (\varphi^*(\omega_Y(S)))^{-1},$$

with the Higgs field:

$$\Theta_A : A^{1,0} \cong A^{0,1} \otimes \varphi^* \omega_Y(S) \subset A^{0,1} \otimes \omega_{Y'}(\varphi^{-1}S) \subset A^{0,1} \otimes \omega_{Y'}(T).$$

Since Y' is ramified over $\hat{S}$ which contains S, the inclusion in the middle holds by Lemma 1.8.

Fact 3.14. *In the maximal case, the Higgs bundle A corresponds to a complex polarized variation of weight 1 and rank 2 on $Y' \backslash \varphi^{-1} S$.*

Proof. In the maximal case (3.9), we get that:

$$\deg \mathcal{C} = \frac{1}{2} \deg(\omega_Y(S)).$$

Therefore,

$$\deg A = \deg A^{1,0} + \deg A^{0,1} = \deg \varphi \cdot (2 \deg \mathcal{C} - \deg(\omega_Y(S))) = 0.$$

Since the degree of A is equal to 0, the polystability of the Higgs bundle G yields a decomposition:

$$G = A \oplus K,$$

where K is a polystable Higgs bundle of degree 0. We note that the complex variation which corresponds to G has the trivial filtration since the monodromies are unipotent, see Lemma 2.12, hence G has the trivial filtration and so A has the trivial filtration. So, the Higgs bundle A is associated to a subvariation $\mathbb{A}$, of weight 1 and rank 2, of the geometric variation of the family

$h : W \to Y'$, by Simpson's correspondence theorem. The complex subvariation $\mathbb{A}$ of the variation $\mathbb{E}_{\mathbb{W}}$ on $Y'\backslash T$ has Higgs field:

$$\Theta_A : A^{1,0} \cong A^{0,1} \otimes \varphi^*(\omega_Y(S)) \subset A^{0,1} \otimes \omega_{Y'}(\varphi^{-1}S) \subset A^{0,1} \otimes \omega_{Y'}(T)$$

which has no poles over $T\backslash\varphi^{-1}S$. So, the complex variation $\mathbb{A}$ is a complex variation on $Y'\backslash\varphi^{-1}S$. Moreover, it is a polarized subvariation. The polarization comes from the standard polarization of the geometric variation $\mathbb{E}_W$ on $Y'\backslash T$. Since $\mathbb{A}$ is a local system on $Y'\backslash\varphi^{-1}S$ with unipotent monodromies around $\varphi^{-1}S$, this polarization can be extended to $Y'\backslash\varphi^{-1}S$. $\square$

Remark 3.16. We note that in general Θ_A is not an isomorphism above the points of the set $\varphi^{-1}(S)$, i.e.

$$\Theta_A : A^{1,0} \cong A^{0,1} \otimes \varphi^*\omega_Y(S) \subset A^{0,1} \otimes \omega_{Y'}(\varphi^{-1}(S)).$$

As we have seen, the last inclusion holds by Lemma 1.8 and it is strict in general, i.e. when $\hat{S} \neq S$. However, note that in the situation when $\Sigma \subset \varphi^*S$ (see above the definition of Σ), or equivalently when $\hat{S} = S$, then $T = \varphi^*S$, hence $Y'\backslash\varphi^*S$ is a Teichmüller curve for the family $h : W \to Y'$. This holds since the subvariation corresponding to the Higgs bundle A has a Higgs field which is an isomorphism, hence the result follows by Theorem 2.22. The case when Σ is the empty set occurs when the divisor D is a smooth divisor. In the following chapter we will give some examples which describe this situation.

Remark 3.17. By (3.10) and Simpson's correspondence theorem, the local system $\mathbb{E}_W$ on $Y'\backslash T$ underlying the geometric variation of the family $h : W \to Y'$ has the decomposition:

$$\mathbb{E}_W = R^1h_*\mathbb{C}_{W\backslash h^{-1}(T)} = \mathbb{E}_Z \oplus \mathbb{A} \oplus \mathbb{K} \oplus \bigoplus_{i=2}^{\nu-1} \mathbb{G}_i, \tag{3.11}$$

where the local system $\mathbb{A}$ corresponds to A, then $\mathbb{K}$ corresponds to K and $\mathbb{E}_Z$ is the local system which corresponds to the geometric variation of the family $g : Z \to Y$.

Let us suppose that g' is the genus of a smooth fiber of the map $h : W \to Y'$. Then as we saw in the course of the proof of Theorem 2.22, the period map $\phi_{\mathbb{A}}$ of the subvariation $\mathbb{A}$ can be factorized as:

$$\begin{tikzcd} \widetilde{Y'\backslash T} \arrow[r] \arrow[rrr, bend left, "\phi_{\mathbb{A}}"] \arrow[rr, bend right, "\phi_{\mathbb{E}_W}"'] & \mathcal{T}_{g'} \arrow[r] & \mathbb{H}_{g'} \arrow[r, "p"] & \mathbb{H} \end{tikzcd}$$

where the composition of the first two maps is the period map $\phi_{\mathbb{E}_W}$ for the geometric variation $\mathbb{E}_W$ of the family $h : W \to Y'$ and the last map p is the projection to the part which corresponds to the complex subvariation $\mathbb{A}$. The derivative of the period map $\phi_{\mathbb{A}}$ can be identified with the Higgs field Θ_A.

Fact 3.15. *In the maximal case, the pullback of the Kodaira-Spencer map Θ_{KS} of the family $f : X \to Y$ can be seen as a direct factor of the Higgs field of the Higgs bundle E_W associated to the family $h : W \to Y'$.*

Proof. As we explained in Remark 3.15

$$\Theta_A = \mathrm{id}_{\varphi^*\mathcal{C}} \otimes \varphi^*\Theta_{KS},$$

where Θ_{KS} is the Kodaira Spencer map of the family $f : X \to Y$. The decompositions:

$$G = A \oplus K$$

and

$$E_W = E_Z \oplus G \oplus \bigoplus_{i=2}^{\nu-1} G_{i,}$$

imply that the Higgs field Θ_A is a direct factor of the Higgs field Θ_G which is a direct factor of the Higgs field of the Higgs bundle associated to the family $h : W \to Y'$ and so we get that the pullback of the Kodaira-Spencer map Θ_{KS} is a direct factor of the Higgs field of the Higgs bundle E_W associated to the family $h : W \to Y'$.

□

Fact 3.16. *In the maximal case one has:*

$$Y \backslash S \cong \mathbb{H}/\Gamma_0,$$

where Γ_0 is the monodromy group of the polarized complex variation $\mathbb{H}_0$ on $Y\backslash S$, where $\mathbb{A} = \varphi^*\mathbb{H}_0$.

Proof. As we explained in Remark 3.15, the Higgs bundle A is the pullback of the Higgs bundle (H_0, Θ_0) on $Y\backslash S$ with Higgs field

$$\Theta_{0|H_0^{1,0}} : H_0^{1,0} \cong H_0^{0,1} \otimes \omega_Y(S),$$

an isomorphism. We recall that $\Theta_0(H^{0,1}) = 0$. In the maximal case one gets:

$$\deg H_0 = 0. \tag{3.12}$$

As the Higgs bundle A corresponds to the complex polarized variation $\mathbb{A}$, $\mathbb{A}$ is the pullback by the finite map φ of the variation $\mathbb{H}_0$ on $Y\backslash S$, which corresponds to the Higgs bundle H_0. Hence, the complex variation $\mathbb{H}_0$ naturally inherits a polarization from the geometric variation $\mathbb{A}$.

One should note that (3.12) and point 3) of Lemma 2.12 imply that the monodromies around points of S of $\mathbb{H}_0$ are unipotent.

Then, bringing all these information together one gets that $\mathbb{H}_0$ is a polarized complex variation of weight 1, rank 2 with corresponding Higgs field an isomorphism, and with unipotent monodromies around the points of S. By Theorem 2.20 one gets:

$$Y \backslash S \cong \mathbb{H}/\Gamma_0,$$

where Γ_0 is the monodromy group of the corresponding representation of $\mathbb{H}_0$.

□

Chapter 4

Examples of Arakelov equality for semistable families of curves uniformized by the unit ball

4.1 Second fundamental form

Let X be a complex manifold and let E be a complex vector bundle on X with a Hermitian metric. We recall that k-valued forms with values in E on X are sections of the bundle

$$\mathcal{A}^k(E) = \bigwedge^k T^*_{\mathbb{C},X} \otimes E,$$

where $T^*_{\mathbb{C},X} = T^*_X \otimes \mathbb{C}$ is the complexified cotangent bundle.

Using the decomposition $T^*_{\mathbb{C},X} = T^{*\,1,0}_X \oplus T^{*\,0,1}_X$, where $T^{*\,1,0}_X$ is isomorphic to the holomorphic cotangent bundle of X, we get the decomposition

$$\mathcal{A}^1(E) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E).$$

Hence, we can decompose any connection $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ on E in two components:

$$\nabla^{1,0} : \mathcal{A}^0(E) \to \mathcal{A}^{1,0}(E) \text{ and } \nabla^{0,1} : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$$

such that $\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$.

Proposition 4.1. *([35],p.177) Let E be a holomorphic vector bundle with a Hermitian metric on a complex manifold X. Then there is a unique connection ∇ such that $\nabla^{0,1} = \overline{\partial}_E$. This connection is called the Chern connection on E.*

Now, we suppose that we have a short exact sequence of holomorphic vector bundles on a complex manifold X :

$$0 \to E_1 \to E \to E_2 \to 0.$$

In general, such a sequence does not split holomorphically. However, the sequence of underlying smooth complex bundles always splits. Indeed, let h be a Hermitian metric on the holomorphic vector bundle E. Then, E_2 is $\mathcal{C}^\infty$-isomorphic to the orthogonal complement of E_1 with respect to the Hermitian metric h, hence one has a $\mathcal{C}^\infty$-splitting:

$$E = E_1 \oplus E_2.$$

Let ∇ be the Chern connection of the metric h on E, i.e. the connection such that $\nabla^{0,1} = \overline{\partial}_E$. On subbundles E_1 and E_2 one has the induced connections ∇_1 and ∇_2 defined by:

$$\nabla_i(s) := p_i(\nabla(s)),$$

where:

- s is any section of E_i, and hence of E;
- p_i is the projection $E_1 \oplus E_2 \to E_i$;

for $i \in \{1, 2\}$.

The connection ∇_1 on E_1 satisfies $\nabla_1^{0,1} = \overline{\partial}_{E_1}$, since E_1 is a holomorphic subbundle of E. We have the commutative diagram:

$$\begin{tikzcd}
\mathcal{A}^0(E_1) \arrow[r, "\nabla_{|\mathcal{A}^0(E_1)}"] \arrow[dr, "\nabla_1"'] & \mathcal{A}^1(E) \arrow[d, "pr"] \\
 & \mathcal{A}^1(E_1)
\end{tikzcd}$$

Now, we define the operator:

$$\beta = \nabla_{|\mathcal{A}^0(E_1)} - \nabla_1 : \mathcal{A}^0(E_1) \to \mathcal{A}^1(E_2).$$

Definition 4.1. The operator β is called the second fundamental form of the subbundle $E_1 \subset E$. It is of type $(1, 0)$ and linear over $\mathcal{C}^\infty$ functions, i.e.

$$\beta \in \mathcal{A}^{1,0}(\mathrm{Hom}(E_1, E_2)).$$

By Theorem 14.3 from [17] the connection matrix of ∇ is

$$\nabla = \begin{pmatrix} \nabla_1 & -\beta^* \\ \beta & \nabla_2 \end{pmatrix},$$

where $\beta^* \in \mathcal{A}^{0,1}(\mathrm{Hom}(E_2, E_1))$ is the adjoint of β and $\overline{\partial}\beta^* = 0$. Hence β^* defines a class

$$[\beta^*] \in H^1(X, \mathrm{Hom}(E_2, E_1)).$$

Proposition 4.2. *([17],Theorem 14.9) The correspondence $E \to [\beta^*]$ induces a bijection from the set of isomorphism classes of extensions of E_1 by E_2 onto the cohomology group $H^1(X, \mathrm{Hom}(E_2, E_1))$. In particular $[\beta^*]$ vanishes if and only if the exact sequence $0 \to E_1 \to E \to E_2 \to 0$ splits holomorphically.*

Definition 4.2. If $i : X_0 \to X$ is an immersion of a complex manifold X_0 into a complex manifold X with a Hermitian metric, then we say that X_0 is totally geodesic if the second fundamental form β_{X_0} of the tangent bundle T_{X_0} vanishes identically on X_0.

If X_0 is a complex submanifold of a compact complex manifold X with a Hermitian metric, then the definition of the normal bundle N_{X_0} is given by the exact sequence:

$$0 \to T_{X_0} \to T_{X|X_0} \to N_{X_0} \to 0.$$

The second fundamental form of the tangent bundle $T_{X_0} \subset T_{X|X_0}$ will be $\beta_{X_0} \in \mathcal{A}^{1,0}(X_0, \mathrm{Hom}(T_{X_0}, N_{X_0}))$.

By Proposition 4.2 we get that for a totally geodesic submanifold X_0, the previous exact sequence splits holomorphically, hence we have a holomorphic isomorphism:

$$T_{X|X_0} \simeq T_{X_0} \oplus N_{X_0}.$$

4.2 Second projective fundamental form

In this section we will give a short review of Mok's paper [53] and it can be considered as the technical background for what follows. Here, we will explain notions such as holomorphic projective connection, second projective fundamental form and tautological foliation on a projectivized tangent bundle. We will see how on a space endowed with a holomorphic projective connection we obtain the projective second fundamental form, using a holomorphic foliation on the projectivized tangent bundle.

Definition 4.3. ([53] §2.1) Let X be a n-dimensional complex manifold for $n > 1$. A holomorphic projective connection Π on X consists of:

- an open covering $\{U_\alpha\}$, with holomorphic coordinates $\{z_1^\alpha, ..., z_n^\alpha\}$;
- holomorphic functions $({}^\alpha\Phi_{ij}^k)_{1\leq i,j,k\leq n}$ on U_α, symmetric in i, j satisfying the trace condition $\sum_k {}^\alpha\Phi_{ik}^k = 0$ for all i and satisfying on $U_{\alpha\beta} = U_\alpha \cap U_\beta$ the transformation rule:

$$^\beta\Phi_{pq}^l = \sum_{i,j,k} {}^\alpha\Phi_{ij}^k \frac{\partial z_i^\alpha}{\partial z_p^\beta} \frac{\partial z_j^\alpha}{\partial z_q^\beta} \frac{\partial z_l^\alpha}{\partial z_k^\beta} + S(f_{\alpha\beta}),$$

where $S(f_{\alpha\beta})$ stands for the Schwarzian derivative of the holomorphic transformation given by the change of variables $z^\alpha = f_{\alpha\beta} z^\beta$.

Definition 4.4. ([53] §2.2) Let $({}^{\alpha}\overline{\Gamma}_{ij}^{k})$ be Riemann-Christoffel symbols of any smooth connection on a complex manifold X. Let Π be a projective connection on X defined as above. We can define a torsion free smooth connection ∇ associated to Π, to be the connection with Riemann-Christoffel symbols:

$$^{\alpha}\Gamma_{ij}^{k} =^{\alpha} \Phi_{ij}^{k} + \frac{1}{n+1}\sum_{l} \delta_{i}^{k}\ {}^{\alpha}\overline{\Gamma}_{lj}^{l} + \frac{1}{n+1}\sum_{l} \delta_{j}^{k}\ {}^{\alpha}\overline{\Gamma}_{il}^{l}.$$

Definition 4.5. ([53] §2.2) Any two smooth connections ∇ and ∇' on a complex manifold X are projectively equivalent if there exists a smooth $(1,0)$-form ω such that:

$$\nabla_u v - \nabla'_u v = \omega(u)v + \omega(v)u,$$

for any smooth $(1,0)$-vector fields u and v on an open set of X.

We assume now that a complex manifold X has a projective connection Π. Let ∇ and ∇' be two smooth torsion-free connections associated to the projective connection on the complex manifold X. Then, ∇ and ∇' are projectively equivalent. Moreover, for any submanifold X_0 of X the second fundamental forms of the tangent bundle of X_0, associated to ∇ and ∇' are the same. As it can be found in [53], the second fundamental form with respect to a torsion-free smooth connection ∇ associated to the projective connection Π is independent of the choice of the background connection $({}^{\alpha}\overline{\Gamma}_{ij}^{k})$ and it is holomorphic.

Remark 4.6. When we refer to the second fundamental form of a submanifold $X_0 \subset X$ we mean the second fundamental form of the holomorphic tangent bundle $T_{X_0} \subset T_{X|X_0}$.

Definition 4.7. The second fundamental form β_{Π} of any submanifold $X_0 \subset X$, with respect to a torsion-free smooth connection ∇ associated to a projective connection Π on X is called the projective second fundamental form of X_0 with respect to Π.

Now, it is plain to see that classes of the complex geodesics submanifolds of X, with respect to smooth torsion-free connections associated to the projective connection Π on the complex manifold X, will be the same.

Definition 4.8. The tautological lifting $\hat{C}$ of some smooth holomorphic curve $C \subset X$ is defined by lifting every point $x \in C$ to the projectivization of the tangent line $[T_xC]$ in $\mathbb{P}T_{X,x}$.

Here we will not give the definition of holomorphic foliation but we refer to [26] for details.

Definition 4.9. A holomorphic foliation of the projectivization of the tangent bundle whose leaves are tautological liftings of holomorphic curves is called a tautological foliation.

Definition 4.10. Complex geodesics on a complex manifold are totally geodesic submanifolds of dimension 1.

Let X be a complex manifold equipped with a tautological foliation $\mathcal{F}$ of the projectivized holomorphic tangent bundle $\mathbb{P}T_X$. Let $\pi : \mathbb{P}T_X \to X$ be the canonical projection. By definition, the leaves of this foliation are 1-dimensional. Now, let $x \in X$ and let $\alpha \in T_x X$. We use the notation $T_{[\alpha]}\mathcal{F} = \mathcal{F}_{[\alpha]}$. Obviously, $d\pi(\mathcal{F}_{[\alpha]}) = \mathbb{C}\alpha$.

The existence of holomorphic projective connections on a complex manifold X is equivalent to the existence of holomorphic foliations on the projectivized holomorphic tangent bundle, by tautological liftings of complex geodesics on X. This relation is described by Proposition 1 from [53]. Here, we will not give the proof of this fact, but we will explain one of its consequences, which is the most important part of this section, the construction of the projective second fundamental form of a submanifold of a complex manifold X, where X is endowed with a tautological foliation, or equivalently with a projective holomorphic connection. This construction can be found in [53] §2.3:

Let X_0 be a complex submanifold of X and let N be the normal bundle of X_0. Let $x_0 \in X_0$ and let $\alpha \in T_{X_0,x_0}$ be a non-zero tangent vector. Let C be a local holomorphic curve on X passing through x_0 and such that $T_{C,x_0} = \mathbb{C}\alpha$, and such that its tautological lifting $\hat{C}$ is a local leaf of the tautological foliation $\mathcal{F}$. Let D be a holomorphic curve on X_0 passing through x_0 and such that $T_{D,x_0} = \mathbb{C}\alpha$. We denote the tautological lifting of D to $\mathbb{P}T_X$ as $\hat{D}$. It is clear that $[\alpha] \in \hat{C} \cap \hat{D}$.

Let $\pi : \mathbb{P}T_X \to X$ be the canonical projection, then we choose the unique vector

$$\eta \in T_{[\alpha]}(\hat{C}) \subset T_{[\alpha]}\mathbb{P}T_{X,x_0},$$

such that

$$d\pi(\eta) = \alpha,$$

and the unique vector

$$\xi \in T_{[\alpha]}(\hat{D}) \subset T_{[\alpha]}\mathbb{P}T_{X,x_0},$$

such that

$$d\pi(\xi) = \alpha.$$

Hence, we get $d\pi(\xi - \eta) = 0$, i.e.

$$\xi - \eta \in \mathrm{Ker}\ d\pi_{[\alpha]}.$$

Let $T_\pi = \mathrm{Ker}\ d\pi$ be the relative tangent bundle of the map $\pi : \mathbb{P}T_X \to X$, see [35] p.95. We define the previous assignment of the vector α to the vector $\xi - \eta$ as the map $\mathcal{A} : T_{X_0,x} \to T_{\pi,[\alpha]}$, which assigns a vector $v \in T_{X_0,x}$ to the vector

$r_v \in T_{\pi,[v]}$ in the same way as α is assigned to $\xi - \eta$. Note that $\mathcal{A}(\alpha) = \xi - \eta$ is independent of the choice of D. Now, we construct the diagram:

$$\begin{tikzcd}
& \pi^*T_X \arrow[d] \arrow[dr] & \\
T\mathbb{P}T_X \arrow[ur, "d\pi"] \arrow[r] \arrow[rr, bend right, "d\pi"'] & \mathbb{P}T_X \arrow[dr, "\pi"'] & T_X \arrow[d] \\
& & X
\end{tikzcd}$$

We have the exact sequence :

$$0 \to T_\pi \to T\mathbb{P}T_X \to \pi^*T_X \to 0.$$

On the other side, using the Euler sequence ([35], p.95) we have :

$$0 \to L \to \pi^*T_X \to T_\pi \otimes L \to 0,$$

where L is the relative tautological bundle on $\mathbb{P}T_{\tilde{X}}$. The previous exact sequence yields the isomorphism :

$$\phi : T_\pi \otimes L \simeq (\pi^*T_X/L).$$

One should note that:

$$(\pi^*T_X)_{[\alpha]} \cong T_{X,x_0},$$

and

$$L_{[\alpha]} \cong T_{D,x_0} \subseteq T_{X_0,x_0}.$$

Hence, we have the canonical projection:

$$\rho : (\pi^*T_X/L)_{[\alpha]} \to (T_X/T_{X_0})_{x_0} \cong (\pi^*N)_{[\alpha]}.$$

Now, we can define

$$B := \rho(\phi(\mathcal{A} \otimes \mathrm{id}_{L_{[\alpha]}})) : L_{[\alpha]} \otimes L_{[\alpha]} \to (\pi^*N)_{[\alpha]},$$

such that:

$$\begin{aligned} B(\alpha \otimes \alpha) &= \rho\left(\phi\left(\mathcal{A} \otimes id_{L_{[\alpha]}}(\alpha \otimes \alpha)\right)\right) \\ &= \rho\left(\phi\left((\xi - \eta) \otimes \alpha\right)\right). \end{aligned}$$

As we mentioned above $\mathcal{A}(\alpha)$ is independent of choice of D, hence $B(\alpha \otimes \alpha)$ is also independent of the choice of D. Moreover, one can see by local considerations that $B(\alpha \otimes \alpha)$ varies holomorphically with α. Hence, one gets an induced holomorphic section

$$\tilde{\sigma} \in H^0(\mathbb{P}T_{X_0}, L^{-2} \otimes \pi^* N),$$

which is defined on fibers by B, and also one has the corresponding section

$$\sigma \in H^0(X_0, S^2 T^*_{X_0} \otimes N).$$

According to [53] §2.3, this section agrees with the projective second fundamental form of X_0 in X with respect to the given holomorphic projective connection on X.

4.3 The complex unit ball

In this section we will define the complex hyperbolic n-space and the complex unit n-ball. We will show that these two Kähler varieties can be identified and then we will give a short review of their basic properties. Then, we will define the tautological foliation on the projectivization of the tangent bundle of the complex unit n-ball. We will end this part by recalling Mok's result which states that the second fundamental form of a submanifold of the quotient of the complex unit n-ball by some discrete subgroup of $\mathrm{PU}(n,1)$, with respect to the Bergman metric, coincides with the projective second fundamental form induced by the tautological foliation.

Definition 4.11. Let $\mathbb{C}^{n,1} = (\mathbb{C}^{n+1}, h)$ be $(n+1)$-dimensional complex space $\mathbb{C}^{n+1}$ with Hermitian form h of signature $(n,1)$, i.e.

$$h(z,w) = z_0\overline{w_0} + z_1\overline{w_1} + \ldots + z_{n-1}\overline{w_{n-1}} - z_n\overline{w_n}.$$

We say that a vector $z \in \mathbb{C}^{n,1}$ is negative if $h(z,z) < 0$.

Definition 4.12. ([27]§3.1) The complex hyperbolic space is the subset of $\mathbb{P}^n$ consisting of negative lines of $\mathbb{C}^{n,1}$. It is naturally biholomorphic to the complex unit n-ball $\mathbb{B}^n$:

$$\mathbb{B}^n = \{z \in \mathbb{C}^n | \, \langle z, z\rangle < 1\},$$

where $\langle ., . \rangle$ is the standard Hermitian product on $\mathbb{C}^n$.

Definition 4.13. The special unitary group $\mathrm{SU}(n,1)$ is the subgroup of the group $\mathrm{SL}(n+1,\mathbb{C})$ of matrices which preserve the Hermitan form h, i.e.

$$\mathrm{SU}(n,1) = \{A \in \mathrm{SL}(n+1,\mathbb{C}) | h(Az, Az) = h(z,z)\}.$$

The projectivization of $\mathrm{SU}(n,1)$ in $\mathrm{PGL}(n+1,\mathbb{C})$ is the group $\mathrm{PU}(n,1)$.

We should note that $\mathrm{PU}(n,1)$ is the group of biholomorphisms of $\mathbb{B}^n$. Moreover, the group $\mathrm{PU}(n,1)$ acts transitively on $\mathbb{B}^n$, see Lemma 3.1.3 from [27], in other words for any $[z],[w] \in \mathbb{B}^n$ there is an element $A \in \mathrm{PU}(n,1)$, such that $A[z] = [w]$.

From now on we will consider that the complex unit ball (or the complex hyperbolic space) is endowed with the Bergman metric, sometimes called the Poincaré metric, whose sectional holomorphic curvature is constant.

Now, let us give the theorem which explains how we get the totally geodesic submanifolds in $\mathbb{B}^n$, which will help us to define the tautological foliation on $\mathbb{P}T_{\mathbb{B}^n}$.

Theorem 4.3. *([27], §3.1.10) Let $F \subset \mathbb{P}^n$ be a complex m-dimensional projective subspace which intersects $\mathbb{B}^n$. Then, $F \cap \mathbb{B}^n$ is a totally geodesic holomorphic submanifold with respect to the Bergman metric, biholomorphically isometric to $\mathbb{B}^m$.*

Hence, by the previous theorem, the complex geodesics on $\mathbb{B}^n$ are obtained as intersections of $\mathbb{B}^n$ with projective lines. On the other side, projective lines are complex geodesics in $\mathbb{P}^n$, with respect to the Fubini-Study metric. We recall that given a point in $\mathbb{B}^n$ and a complex tangent line at this point, there is a unique complex geodesic trough that point tangent to the complex tangent line. Now we will define the tautological foliation of the tangent bundles of the projective space $\mathbb{P}^n$ and $\mathbb{B}^n$.

Definition 4.14. ([53]§2.3) The tautological foliation on the projectivization of the tangent bundle of the projective space $\mathbb{P}^n$ is defined by the tautological lifting of projective lines. The tautological foliation on the projectivization of the tangent bundle of $\mathbb{B}^n$ is defined by tautological liftings of restrictions of projective lines to the complex hyperbolic space $\mathbb{B}^n$.

Definition 4.15. A complex hyperbolic space form of dimension n is a quotient of the complex hyperbolic n-space $\mathbb{B}^n$ by some torsion-free discrete subgroup Γ of the group of holomorphic automorphisms $\mathrm{PU}(n,1)$. These quotients are also called ball quotients.

The group $\mathrm{PGL}(\mathbb{C}, n+1)$ is the group of automorphisms of $\mathbb{P}^n$ and then the tautological foliation on $\mathbb{P}T_{\mathbb{P}^n}$ is invariant under the action of this group. One has to note that the tautological foliation on $\mathbb{P}T_{\mathbb{B}^n}$ descends also to the tautological foliation on the projectivization of the tangent bundle of any complex hyperbolic space form. This holds since $\mathrm{PU}(n,1)$, the group of holomorphic automorphisms of $\mathbb{B}^n$, is a subgroup of the projective linear group $\mathrm{PGL}(\mathbb{C}, n+1)$.

Definition 4.16. The holomorphic projective connection which corresponds to the tautological foliation on $\mathbb{P}T_{\mathbb{B}^n}$, or to the tautological foliation on the projectivization of the tangent bundle of a complex hyperbolic space form is called the canonical holomorphic projective connection. The projective second fundamental form of any holomorphic immersion, with respect to this connection is called the canonical projective second fundamental form.

Also, let us note that a complex hyperbolic space form is endowed with the Bergman metric, since the Bergman metric is $\mathrm{PU}(n,1)$-invariant, see Chapter 4 Proposition 2 in [54] .

Let us finish this section by giving Lemma 1 and Lemma 2 from [53], which describe the second fundamental form of holomorphic immersions into a complex hyperbolic space form, with respect to the Bergman metric.

Lemma 4.4. *Let (X, g) be a complex hyperbolic space form endowed with the Bergman metric. The smooth connection of the Bergman metric g is associated to the canonical holomorphic projective connection on X. Let $i : X_0 \to X$ be a holomorphic immersion, and denote by β the $(1,0)$-part of the second fundamental form of X_0, with respect to the Bergman metric g. Then β is holomorphic. Moreover, the second fundamental form on X_0 with respect to the Bergman metric g agrees with the canonical projective second fundamental form of X_0 in X.*

4.4 Arakelov inequality and quotients of the complex 2-ball

In this section we will show that the direct image of the pluricanonical relative sheaf of a semistable family of curves, uniformized by the complex unit 2-ball, over a curve, contains an invertible subsheaf which satisfies the maximal case in Arakelov inequality, under the assumption that all singular fibers of the family are totally geodesic.

Throughout this section we will suppose that $f : X \to Y$ is a projective family of curves such that:

- the genus of Y is at least 2;
- the family is smooth over $U = Y \backslash S$, where S is a finite set of points on Y;
- $X \simeq \mathbb{B}^2/\Gamma$ is a quotient of the complex 2-ball by a torsion-free discrete cocompact subgroup Γ of $\mathrm{PU}(2,1)$, i.e. X is a two dimensional complex hyperbolic space form.

We will denote by $g : \mathbb{B}^2 \to \mathbb{D}$, where $\mathbb{D} \cong \mathbb{H}$, a lift of the map f at the level of the universal cover. For a local system of coordinates $z = (x, y)$ on $\mathbb{B}^2$ we will use the notations: $g_x = \dfrac{\partial g}{\partial x}, g_y = \dfrac{\partial g}{\partial y}$.

Let $i : C \to X$ be a smooth fiber of the family:

$$C = f^{-1}(y), y \in U = Y \backslash S.$$

The second fundamental form of the fiber C with respect to the Bergman metric on X is:

$$\beta_C \in \mathcal{A}^{1,0}(C, \mathrm{Hom}(T_C, N_C)).$$

Since $i : C \hookrightarrow X$ is a holomorphic embedding, by Lemma 4.4 β_C is holomorphic and one gets:

$$\beta_C \in \Gamma(C, \omega_C^{\otimes 2} \otimes N_C).$$

Moreover, by Lemma 4.4 the second fundamental form β_C coincides with the canonical projective second fundamental form of C in X, induced by the tautological foliation on the projectivization of the tangent space of $\mathbb{B}^2$.

Lemma 4.5. *For the canonical projective second fundamental form of C in X, one has: $\beta_C \in \Gamma(C, \omega_C^{\otimes 2})$.*

Proof. This is the consequence of the fact that the normal bundle of a smooth fiber in any family is trivial. Let us explain this. We have $C = f^{-1}(y), y \in U$. Then,

$$(f^*T_Y)_{|C} = C \times (T_Y)_y$$

is a trivial bundle. On the other hand, for every $x \in C$ the map $df_x : (T_{X|C})_x \to (f^*T_Y)_x$ is a surjective map, since f is surjective and C is smooth. The kernel of that map is $(T_C)_x$ and

$$(f^*T_Y)_x \cong (T_{X|C})_x/(T_C)_x.$$

Using the fact that

$$N_C \simeq T_{X|C}/T_C,$$

we have $N_C \simeq (f^*T_Y)_{|C}$, hence N_C is trivial. Finally, we get that for a smooth fiber C from the family, the second fundamental form $\beta_C \in \Gamma(C, \omega_C^{\otimes 2})$. $\square$

The second fundamental form β_C gives rise to a holomorphic section since f is holomorphic:

$$\beta \in H^0(X \backslash \mathrm{Sing}(f), \omega_{X/Y}^{\otimes 2} \otimes f^*T_Y).$$

Since the set of singular points on fibers of the family has dimension 0, by Hartogs theorem β can be extended to a section:

$$\beta \in H^0(X, \omega_{X/Y}^{\otimes 2} \otimes f^*T_Y). \tag{4.1}$$

Definition 4.17. The section $\beta \in H^0(X, \omega_{X/Y}^{\otimes 2} \otimes f^*T_Y)$ is called the second fundamental form of the family $f : X \to Y$. Those fibers along which β vanishes are totally geodesic fibers.

Definition 4.18. Let D be a possibly singular curve in $\mathbb{B}^2$. The curve D is said to be totally geodesic in $\mathbb{B}^2$ if its irreducible components are smooth and totally geodesic in $\mathbb{B}^2$. Let $\tau : \mathbb{B}^2 \to X$ be the universal covering map. Let X_s be a singular fiber in the family $f : X \to Y$. The fiber X_s is totally geodesic in X if $\tau^{-1}(X_s)$ is totally geodesic in $\mathbb{B}^2$.

The following lemma states that totally geodesic fibers in the family can only be singular fibers. The main technical tool for this lemma will be §1 from [53].

Lemma 4.6. *Totally geodesic fibers in the family $f : X = \mathbb{B}^2/\Gamma \to Y$ can't be smooth fibers.*

Proof. According to Hirzebruch's proportionality theorem the Chern numbers of $X = \mathbb{B}^2/\Gamma$ are proportional with strictly positive proportionality factor to the Chern numbers of $\mathbb{P}^2$, since $\mathbb{P}^2$ is the compact dual of $\mathbb{B}^2$. The proportionality factor is equal to the volume of X. The first Chern class of $\mathbb{P}^2$ is:

$$[c_1(\mathbb{P}^2)] = 3[\xi],$$

where $[\xi] \in H^2(\mathbb{P}^2, \mathbb{Z})$ is the Poincaré dual to the hyperplane divisor in $\mathbb{P}^2$.

On the other side, we have $c_1(X) = -3\xi'$, where ξ' is the normalized metric form of the Bergman metric on X, with constant negative sectional curvature -2. By [9] §1, for any submanifold C on X, we have :

$$c_1(C) = -(2\eta + \sigma),$$

where $\sigma = \text{tr } (i\beta \wedge \beta^*)$ is a $(1,1)$-real form which vanishes if C is totally geodesic and η is the restriction of ξ' to C, i.e. $\eta = \xi'_{|C}$. Let C be a totally geodesic fiber of the family $f : X \to Y$. Then, we have:

$$c_1(C) = -2\eta.$$

Moreover, if we suppose that C is a smooth fiber, then by the adjunction formula one has:

$$\omega_C = (\omega_X \otimes \mathcal{O}_C(C))_{|C} = \omega_{X|C},$$

since the normal bundle of the fiber $N_C = \mathcal{O}_C(C)$ is trivial. This yields that:

$$c_1(\omega_C) = c_1(\omega_{X|C}),$$

then for the dual bundles, the tangent bundles we get

$$c_1(C) = c_1(X)_{|C},$$

and

$$-2\eta = -3\eta.$$

Hence, $\eta = 0$. This is not possible since η is the pullback of the Kähler form on X of constant negative sectional curvature. Therefore, the assumption that C is a smooth fiber is false. So, totally geodesic fibers in the family, if any, can only be singular fibers. $\square$

Before we give the most important results of this section, let us recall several well known details about line bundles on the projective line:

- We recall the definition of the tautological bundle on $\mathbb{P}^1$. It is defined as:

  $$L = \{([z], \xi) \in \mathbb{P}^1 \times \mathbb{C}^2 | \xi \in \mathbb{C} \cdot z\} \cong \mathcal{O}(-1),$$

  with fibers

  $$L_{[z]} = \mathbb{C} \cdot z, z \neq 0.$$

  The transition functions of this line bundle are:

  $$l_{01}(z) = \frac{z_0}{z_1} \text{ and } l_{10}(z) = \frac{z_1}{z_0}.$$

- The dual bundle of $\mathcal{O}(-1)$ is denoted as $\mathcal{O}(1)$ with transition functions:

  $$t^1_{01}(z) = \frac{z_1}{z_0} \text{ and } t^1_{10}(z) = \frac{z_0}{z_1}$$

  which are obtained as dual (transposed) maps of the transition functions of the line bundle $\mathcal{O}(-1)$. It is well known, that all line bundles on $\mathbb{P}^1$ are isomorphic to $\mathcal{O}(-m) = \mathcal{O}(-1)^{\otimes m}$ or to $\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}$, for m a non-negative integer, with transition functions equal to the m-th power of the transition functions of $\mathcal{O}(-1)$ or $\mathcal{O}(1)$.

- An easy calculation shows that the transition functions of the tangent bundle on $\mathbb{P}^1$ are:

  $$t_{01}(z) = -\left(\frac{z_1}{z_0}\right)^2 \text{ and } t_{10}(z) = -\left(\frac{z_0}{z_1}\right)^2.$$

  It is well known that two holomorphic vector bundles of rank r over some complex manifold are isomorphic if and only if there exists an open covering $\{U_\alpha\}$ on that manifold relative to which their cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ given by transition functions are equivalent, in the sense that there exist holomorphic maps:

  $$\lambda_\alpha : U_\alpha \to \mathrm{GL}(r, \mathbb{C})$$

  such that

  $$g'_{\alpha\beta} = \lambda_\alpha g_{\alpha\beta} \lambda_\beta^{-1}, \text{in } U_\alpha \cap U_\beta.$$

  If we define the maps, $\lambda_0 : U_0 \to \mathbb{C}$ as $\lambda_0(z) = 1$ and $\lambda_1 : U_1 \to \mathbb{C}$ as $\lambda_1(z) = -1$, then the transition functions of the tangent bundle of $\mathbb{P}^1$ become:

  $$t_{01}(z) = \left(\frac{z_1}{z_0}\right)^2 \text{ and } t_{10}(z) = \left(\frac{z_0}{z_1}\right)^2, \text{in } U_0 \cap U_1,$$

  therefore the tangent bundle of $\mathbb{P}^1$ is isomorphic to the line bundle $\mathcal{O}(2)$.

Lemma 4.7. *One has an isomorphism:*

$$T_{\mathbb{P}^1} \otimes L \simeq \left(\mathbb{P}^1 \times \mathbb{C}^2\right)/L,$$

where $T_{\mathbb{P}^1}$ *is the tangent bundle and* L *is the tautological bundle of* $\mathbb{P}^1$.

Proof. Let $\{z_0, z_1\}$ be a system of local coordinates on $\mathbb{C}^2$, then on $\mathbb{P}^1$ we choose the standard open covering by sets:

$$U_0 = \{[z] = [z_0 : z_1] | z_0 \neq 0\} \text{ and}$$

$$U_1 = \{[z] = [z_0 : z_1] | z_1 \neq 0\}.$$

Let $u_i = \dfrac{z_j}{z_i}, i \neq j$ be the local coordinate on U_i and let us give the trivialisation maps for the bundles $T_{\mathbb{P}^1} \otimes L$ and $(\mathbb{P}^1 \times \mathbb{C}^2)/L$. We suppose that

$$[z] = [z_0 : z_1] \in U_i,$$

then we have :

$$(T_{\mathbb{P}^1} \otimes L)_{[z]} = \mathbb{C}\{\frac{\partial}{\partial u_i} \otimes z\}$$

and the trivialisation map is given by

$$\frac{\partial}{\partial u_i} \otimes \ z \mapsto z_i.$$

For the vector bundle $(\mathbb{P}^1 \times \mathbb{C}^2)/L$ we have:

$$\left(\left(\mathbb{P}^1 \times \mathbb{C}^2\right)/L\right)_{[z]} \simeq \mathbb{C}^2/\mathbb{C} \cdot z,$$

then:

- if $[z] \in U_0$, representatives of the vectors from $((\mathbb{P}^1 \times \mathbb{C}^2)/L)_{[z]}$ are of the form $\begin{pmatrix} 0 \\ \mu \end{pmatrix}$ and the trivialisation map is:

$$\begin{pmatrix} 0 \\ \mu \end{pmatrix} \mapsto \mu;$$

- if $[z] \in U_1$, representatives of the vectors from $((\mathbb{P}^1 \times \mathbb{C}^2)/L)_{[z]}$ are of the form $\begin{pmatrix} \nu \\ 0 \end{pmatrix}$ and the trivialisation map is:

$$\begin{pmatrix} \nu \\ 0 \end{pmatrix} \mapsto \nu.$$

Let $\phi : T_{\mathbb{P}^1} \otimes L \to (\mathbb{P}^1 \times \mathbb{C}^2)/L$ be the map defined by:

$$\phi_{|U_0}\left(\frac{\partial}{\partial u_0} \otimes z\right) = \begin{pmatrix} 0 \\ z_0 \end{pmatrix}, [z] \in U_0;$$

$$\phi_{|U_1}\left(\frac{\partial}{\partial u_1} \otimes \ z\right) = \begin{pmatrix} z_1 \\ 0 \end{pmatrix}, [z] \in U_1.$$

We will show that the map ϕ is well defined. Let us suppose that $[z] \in U_0 \cap U_1$, then a tangent vector $r = \dfrac{\partial}{\partial u_0}$ at the point $[z]$ can be seen as:

$$r = -\left(\frac{z_0}{z_1}\right)^2 \frac{\partial}{\partial u_1},$$

using the transition maps of the tangent bundle $T_{\mathbb{P}^1}$, defined as above.
Then, for $[z] \in U_0 \cap U_1$ one has:

$$v_0 = \phi_{|U_0}(r \otimes z) = \begin{pmatrix} 0 \\ z_0 \end{pmatrix};$$

$$v_1 = \phi_{|U_1}(r \otimes z) = \phi_{|U_1}\left(-\left(\frac{z_0}{z_1}\right)^2 \frac{\partial}{\partial u_1} \otimes z\right) = \begin{pmatrix} -\left(\frac{z_0}{z_1}\right)^2 z_1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{z_0^2}{z_1} \\ 0 \end{pmatrix}.$$

Let us show that the vectors v_0 and v_1 are equal in the fiber $((\mathbb{P}^1 \times \mathbb{C}^2)/L)_{[z]}$. By the following computation:

$$v_1 = \begin{pmatrix} -\frac{z_0^2}{z_1} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ z_0 \end{pmatrix} - \frac{z_0}{z_1}\begin{pmatrix} z_0 \\ z_1 \end{pmatrix},$$

we get that vectors v_0 and v_1 are equal in the quotient space $((\mathbb{P}^1 \times \mathbb{C}^2)/L)_{[z]}$. This implies that the map ϕ is well defined. Besides that, it is an injective and surjective map, so we have an explicitly described isomorphism between the bundles $T_{\mathbb{P}^1} \otimes L$ and $(\mathbb{P}^1 \times \mathbb{C}^2)/L$. □

In section 4.2, following the result of Mok we explained that it is possible to define the second fundamental form of a holomorphic immersion into $X = \mathbb{B}^2/\Gamma$ only using the tautological foliation of $\mathbb{P}T_X$, without any references to the affine connection. Here, following that idea which was presented at the end of section 4.2, we will explicitly describe the second fundamental form of the projective family

$$f : X = \mathbb{B}^2/\Gamma \to Y,$$

at the level of the universal cover $\mathbb{B}^2$.

Lemma 4.8. *Let $g : \mathbb{B}^2 \to \mathbb{D}$ be a lift of the map $f : X = \mathbb{B}^2/\Gamma \to Y$ at the level of the universal covering. The second fundamental form of fibers of the family $g : \mathbb{B}^2 \to \mathbb{D}$ can be identified with the function:*

$$F(z) = g_y^2(z)g_{xx}(z) - 2g_x(z)g_y(z)g_{xy}(z) + g_x^2(z)g_{yy}(z),$$

in the system of coordinates $z = (x, y)$ on $\mathbb{B}^2$.

More precisely, let $\tilde{\beta} \in H^0(\mathbb{B}^2, \omega_{\mathbb{B}^2/\mathbb{D}}^{\otimes 2} \otimes g^*T_{\mathbb{D}})$ be the second fundamental form of the family $g : \mathbb{B}^2 \to \mathbb{D}$. In the trivilization of the relative tangent bundle $T_{\mathbb{B}^2/\mathbb{D}}$ given by

$$z \mapsto s(z) = \begin{pmatrix} g_y(z) \\ -g_x(z) \end{pmatrix}$$

the function F is defined by

$$F(z) = \tilde{\beta}_z\left(\begin{pmatrix} g_y(z) \\ -g_x(z) \end{pmatrix} \otimes \begin{pmatrix} g_y(z) \\ -g_x(z) \end{pmatrix}\right) \in (g^*T_{\mathbb{D}})_z \cong \mathbb{C}.$$

Proof. **(A)** The tangent bundle of $\mathbb{B}^2$ is trivial, i.e. $T_{\mathbb{B}^2} = \mathbb{B}^2 \times \mathbb{C}^2$. The projectivization of the tangent bundle is $\mathbb{P}T_{\mathbb{B}^2} = \mathbb{B}^2 \times \mathbb{P}^1$. The map $\pi : \mathbb{P}T_{\mathbb{B}^2} \to \mathbb{B}^2$ is the canonical projection. Let $T_\pi = \operatorname{Ker} d\pi$. It is plain to see that: $T_\pi \cong T_{\mathbb{P}^1}$.

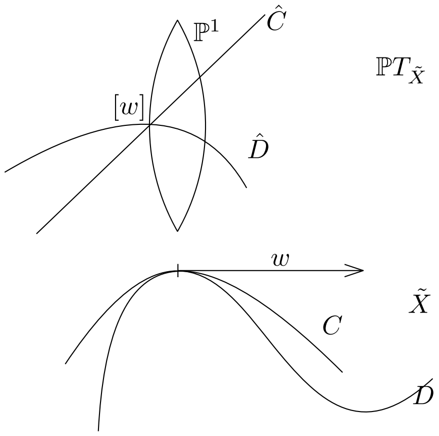
Let $D = g^{-1}(c), c \in \mathbb{D}$. Let $z_0 = (x_0, y_0) \in D$ be a smooth point in the fiber D. We suppose that for the points z around z_0, i.e. for $z \in D_{z_0} = B(z_0, \epsilon) \cap D$ for ϵ small enough, one has $g_y(z) \neq 0$. Then, a trivilization of $T_{D_{z_0}}$ is given by the section:

$$z \mapsto s(z) = \begin{pmatrix} g_y(z) \\ -g_x(z) \end{pmatrix}.$$

The implicit function theorem provides a holomorphic parametrization of the curve D near the fixed point z_0, given by:

$$\lambda \mapsto (x(\lambda), y(\lambda)), \text{ where } g(x(\lambda), y(\lambda)) = c,$$

and such that $x(0) = x_0, y(0) = y_0, \dot{x}(0) = g_y(z_0)$ and $\dot{y}(0) = -g_x(z_0)$.



The tautological lifting of D_{z_0} to the space $\mathbb{P}T_{\mathbb{B}^2}$ is defined by the lifting of a point $z \in D_{z_0}$ to the point $[s(z)] \in \mathbb{P}T_{\mathbb{B}^2,z} \cong \mathbb{P}^1$. More precisely, the tautological lifting of D_{z_0} is defined as the curve:

$$\hat{D}_{z_0} = \{(z, [s(z)]), z \in D_{z_0}\} \subset D_{z_0} \times \mathbb{P}^1 \subset \mathbb{P}T_{\mathbb{B}^2},$$

i.e. $\hat{D}_{z_0}$ as the graph of the function :

$$z \in D_{z_0} \mapsto [s(z)] = [g_y(z) : -g_x(z)],$$

and since we supposed that $g_y(z) \neq 0$ then

$$z \in D_{z_0} \mapsto [s(z)] = \left[1 : -\frac{g_x(z)}{g_y(z)}\right] \in U_0,$$

where $\{U_0, U_1\}$ is the standard open covering on $\mathbb{P}^1$. So, we get the local description of the curve $\hat{D}_{z_0}$ by the parametrization:

$$\lambda \mapsto \left(x(\lambda), y(\lambda), [1 : -\frac{g_x(x(\lambda), y(\lambda))}{g_y(x(\lambda), y(\lambda))}]\right),$$

or to simplify notation, the parametrization of $\hat{D}_{z_0}$ is given as:

$$\lambda \mapsto \left(x(\lambda), y(\lambda), -\frac{g_x(x(\lambda), y(\lambda))}{g_y(x(\lambda), y(\lambda))}\right).$$

(B) Let $w = s(z_0) = \begin{pmatrix} g_y(z_0) \\ -g_x(z_0) \end{pmatrix}$ be a tangent vector of D_{z_0} at the point z_0. Let C be a geodesic on $\mathbb{B}^2$ intersecting D_{z_0} at the point z_0, such that w is a common tangent vector for both curves. Being a geodesic, C is a line on the ball in the direction of the vector w. We have the following parametrization for C :

$$\lambda \mapsto (\lambda g_y(z_0), -\lambda g_x(z_0)).$$

The tautological lifting $\hat{C}$ of the curve C is a leaf of tautological foliation on $\mathbb{P}T_{\mathbb{B}^2}$ and it intersects $\hat{D}_{z_0}$ at the point $[w]$. We can describe the curve $\hat{C}$ by the parametrization :

$$\lambda \mapsto \left(\lambda g_y(z_0), -\lambda g_x(z_0), \left[1 : -\frac{g_x(z_0)}{g_y(z_0)}\right]\right),$$

or

$$\lambda \mapsto \left(\lambda g_y(z_0), -\lambda g_x(z_0), -\frac{g_x(z_0)}{g_y(z_0)}\right).$$

Using the parametrization introduced above, a tangent vector of $\hat{D}_{z_0}$ at the point $[w] \in \mathbb{P}T_{\mathbb{B}^2,z}$ is :

$$\xi = \left(\dot{x}(\lambda), \dot{y}(\lambda), \frac{\partial}{\partial\lambda}\left(-\frac{g_x(x(\lambda), y(\lambda))}{g_y(x(\lambda), y(\lambda))}\right)\right)_{|\lambda=0} \in T_{[w]}(\hat{D}_{z_0}) \subseteq T_{[w]}\mathbb{P}T_{\mathbb{B}^2}.$$

Using that:

- $\dot{x}(0) = g_y(z_0), \quad \dot{y}(0) = -g_x(z_0),$
- $\dfrac{\partial}{\partial\lambda}(g_x(x(\lambda), y(\lambda))) = g_{xx}\dot{x} + g_{xy}\dot{y},$
- $\dfrac{\partial}{\partial\lambda}(g_y(x(\lambda), y(\lambda))) = g_{yy}\dot{y} + g_{xy}\dot{x},$

we get:

$$\begin{aligned}\frac{\partial}{\partial\lambda}\left(-\frac{g_x(x(\lambda), y(\lambda))}{g_y(x(\lambda), y(\lambda))})\right)_{|\lambda=0} &= \left(-\frac{(g_{xx}\dot{x} + g_{xy}\dot{y})g_y - g_x(g_{xy}\dot{x} + g_{yy}\dot{y})}{g_y^2}\right)_{|\lambda=0} \\ &= -g_{xx}(z_0) + 2\frac{g_x(z_0)}{g_y(z_0)}g_{xy}(z_0) - \left(\frac{g_x(z_0)}{g_y(z_0)}\right)^2 g_{yy}(z_0),\end{aligned}$$

and hence

$$\xi = \left(g_y(z_0), -g_x(z_0), -g_{xx}(z_0) + 2\frac{g_x(z_0)}{g_y(z_0)}g_{xy}(z_0) - \left(\frac{g_x(z_0)}{g_y(z_0)}\right)^2 g_{yy}(z_0)\right),$$

whilst a tangent vector of $\hat{C}$ at the point $[w]$ is

$$\eta = (g_y(z_0), -g_x(z_0), 0) \in T_{[w]}(\hat{C}) \subseteq T_{[w]}\mathbb{P}T_{\mathbb{B}^2}.$$

We get the vector :

$$\eta - \xi = \left(0, 0, g_{xx}(z_0) - 2\frac{g_x(z_0)}{g_y(z_0)}g_{xy}(z_0) + \left(\frac{g_x(z_0)}{g_y(z_0)}\right)^2 g_{yy}(z_0)\right).$$

Therefore, we get $d\pi(\eta - \xi) = 0$, i.e.

$$\eta - \xi \in T_{\pi,[w]}.$$

As we explained in the end of Section 4.2, in order to define the the second fundamental form of D_{z_0} which is a submanifold of dimension 1 on the complex unit 2-ball, which is equipped with the projective connection, we have to assign to the vector w, the unique vector, $r_w = \mathcal{A}(w)$ from the kernel of the map π. Summarising what we have seen so far we get:

$$r_w = \eta - \xi.$$

As $\eta - \xi \in T_{\pi,[w]}$ and $[w] = \left[1 : -\dfrac{g_x(0)}{g_y(0)}\right] \in U_0$, then in the local coordinates on $\mathbb{P}(T_{\mathbb{B}^2,z_0}) \cong \mathbb{P}^1$ we can see the vector $\eta - \xi$ as the vector :

$$r_w = \left(g_{xx}(z_0) - 2\frac{g_x(z_0)}{g_y(z_0)}g_{xy}(z_0) + \left(\frac{g_x(z_0)}{g_y(z_0)}\right)^2 g_{yy}(z_0)\right)\frac{\partial}{\partial u_0},$$

where $u_i = \frac{v_j}{v_i}$, $i \neq j$, is a local coordinate on the standard open set U_i of $\mathbb{P}(T_{\mathbb{B}^2,z_0})$ and $\{v_0, v_1\}$ are local coordinates on $T_{\mathbb{B}^2,z_0}$.

(C) Let us make a short review of what we saw in the end of Section 4.2 but here applied to the case of the complex unit 2-ball. Following the notations from Section 4.2 we have the map:

$$\mathcal{A} : T_{D_{z_0}} \to T_{\pi,[v]},$$

or using the isomorphism $T_{D_{z_0},z} \cong L_{[v]}$ we have the map

$$\mathcal{A} : L_{[v]} \to T_{\pi,[v]},$$

$$\mathcal{A}(v) = r_v,$$

where $T_{D_{z_0},z} = \mathbb{C}v$, at a point $z \in D_{z_0}$. This map gives rise to the map $\tilde{\sigma}_{[v]}$ which will induce the second fundamental form of D_{z_0}:

$$\tilde{\sigma}_{[v]} : L_{[v]} \otimes L_{[v]} \to T_{\pi[v]} \otimes L_{[v]} \to (\pi^* T_{\mathbb{B}^2}/L)_{[v]},$$

which is defined as:

$$\tilde{\sigma}_{[v]} = \phi(\mathcal{A} \otimes \mathrm{id}_{L_{[v]}}).$$

In the case of the complex unit 2-ball, using the isomorphism ϕ constructed in Lemma 4.7, $\tilde{\sigma}$ will be defined as:

$$\tilde{\sigma}_{[v]}(v,v) = \phi(\mathcal{A} \otimes \mathrm{id}_L(v,v)) = \phi(r_v \otimes v),$$

$$\tilde{\sigma}_{[v]}(v,v) = \begin{cases} \begin{pmatrix} 0 \\ v_0\alpha \end{pmatrix}, & \text{if } [v] \in U_0 \text{ and } r_v = \alpha\frac{\partial}{\partial u_0}; \\ \begin{pmatrix} v_1\beta \\ 0 \end{pmatrix}, & \text{if } [v] \in U_1 \text{ and } r_v = \beta\frac{\partial}{\partial u_1}. \end{cases}$$

The morphism $\tilde{\sigma}_{[v]}$ gives rise to the holomorphic section:

$$\tilde{\sigma} \in H^0(\mathbb{P}T_{D_{z_0}}, L^{-2} \otimes \pi^* T_{\mathbb{B}^2}/L).$$

Note that the isomorphism $L_{[v]} \cong T_{D_{z_0},z}$ implies:

$$\pi^* T_{\mathbb{B}^2}/L \cong \pi^* N_{D_{z_0}}.$$

Hence,

$$\tilde{\sigma} \in H^0(\mathbb{P}T_{D_{z_0}}, L^{-2} \otimes \pi^* N_{D_{z_0}}).$$

At the end we will apply the isomorphism $dg : N_{D_{z_0},z} \cong T_{\mathbb{D},g(z)} \cong \mathbb{C}$, given by the linear operator $dg_{(z)} \begin{pmatrix} \lambda \\ \gamma \end{pmatrix} = (g_x, g_y)_{(z)} \begin{pmatrix} \lambda \\ \gamma \end{pmatrix} = g_x(z)\lambda + g_y(z)\gamma$ and we get the holomorphic section:

$$\sigma \in H^0(D_{z_0}, T^{-2}_{D_{z_0}} \otimes g^* T_{\mathbb{D}})$$

which is the second fundamental form of D_{z_0}. This section on fibers is given by:

$$\sigma_z = dg_{(z)}(\tilde{\sigma}_{[v]}) : T_{D_{z_0},z} \otimes T_{D_{z_0},z} \to \mathbb{C},$$

such that:

$$\sigma_z(v,v) = \begin{cases} v_0 \alpha g_y(z) \text{ if } \ [v] \in U_0 \text{ and } r_v = \alpha \frac{\partial}{\partial u_0}; \\ v_1 \beta g_x(z) \text{ if } [v] \in U_1 \text{ and } r_v = \beta \frac{\partial}{\partial u_1}. \end{cases}$$

Now, applying what we have seen so far, to the vector w at point z_0 on D_{z_0}, with

$$r_w = \left(g_{xx}(z_0) - 2\frac{g_x(z_0)}{g_y(z_0)} g_{xy}(z_0) + \left(\frac{g_x(z_0)}{g_y(z_0)} \right)^2 g_{yy}(z_0) \right) \frac{\partial}{\partial u_0},$$

we get:

$$\begin{aligned} \sigma_{z_0}(w,w) &= g_y(z_0) \left(g_{xx}(z_0) - 2\frac{g_x(z_0)}{g_y(z_0)} g_{xy}(z_0) + \left(\frac{g_x(z_0)}{g_y(z_0)} \right)^2 g_{yy}(z_0) \right) g_y(z_0) \\ &= g_y^2(z_0) g_{xx}(z_0) - 2g_x(z_0)g_y(z_0)g_{xy}(z_0) + g_x^2(z_0)g_{yy}(z_0), \end{aligned}$$

which is the expression for the second fundamental form of D_{z_0} in the local trivilization, i.e. $\sigma_z = \tilde{\beta}_z$. Since the last expression is symmetric in g_x and g_y, the condition $g_y(z) \neq 0$ did not play any role.

Moreover, the last expression is well defined at every point of the fiber D, therefore the extension of the second fundamental form of the fiber D can be identified with the function

$$F : \mathbb{B}^2 \to \mathbb{C},$$

where

$$F(z) = g_y^2(z) g_{xx}(z) - 2g_x(z)g_y(z)g_{xy}(z) + g_x^2(z)g_{yy}(z).$$

□

Lemma 4.9. *In the previous notations, let the family $f : X \to Y$ be a semistable family of curves, then the function $F : \mathbb{B}^2 \to \mathbb{C}$ vanishes at order exactly one on the totally geodesic fibers, if any, in the family $g : \mathbb{B}^2 \to \mathbb{D}$.*

Proof. The family $f : X \to Y$ is a semistable family of curves, hence the singular fibers of the family $g : \mathbb{B}^2 \to \mathbb{D}$ are reduced normal crossing divisors.

By the previous lemma, the second fundamental form of a fiber at a point z in the family $g : \mathbb{B}^2 \to \mathbb{D}$ is:

$$F(z) = g_y^2(z)g_{xx}(z) - 2g_x(z)g_y(z)g_{xy}(z) + g_x^2(z)g_{yy}(z).$$

Note that the map F is well defined everywhere on $\mathbb{B}^2$, including singular points of the family $\mathrm{Sing}(g)$. Let D be a singular and totally geodesic fiber in the family $g : \mathbb{B}^2 \to \mathbb{D}$. Since $\mathrm{PU}(2,1)$ acts transitively on $\mathbb{B}^2$, we can suppose, after some change of coordinates on the ball, that:

- $D = g^{-1}(0)$;
- $(0,0) \in D$ and it is a singular point for the map g.

Since any singular point of the fiber D is a normal crossing of two branches and D is a geodesic on the ball, we can suppose that the branches of D which intersect at the point $(0,0)$ are the branches $\{y = 0\}$ and $\{x - cy = 0\}, c \in \mathbb{C}$. Recall that the second fundamental form is invariant under the action of $\mathrm{PU}(2,1)$.

One has $g(cy, y) = 0$ and $g(x, 0) = 0$. Since D is totally geodesic, on the branch $\{y = 0\}$ we have:

$$F(x, 0) = 0.$$

We write the function g as:

$$g(x, y) = \sum_{i,j} a_{ij} x^i y^j,$$

where a_{ij} are Taylor's coefficients in the neighborhood of the point $(0,0)$. The equation $g(x, 0) = 0$ gives:

$$a_{k0} = 0, \text{ for all } k.$$

The fact that $(0,0)$ is a singular point yields:

$$a_{01} = 0.$$

On the other side, for the function g in the neighborhood of the point $(0,0)$ one has:

$$g(x, y) = ay(x - cy) + \sum_{i+j \geq 3} a_{i,j} x^i y^j,$$

with $a \neq 0$ since D is nodal at $(0,0)$.

So, we get:

$$a_{11} = \frac{\partial^2 g}{\partial x \partial y}(0,0) = a \neq 0. \tag{4.2}$$

We have:

$$\begin{aligned} F(x,y) &= (\sum_{j\geq 1} j a_{ij} x^i y^{j-1})^2 (\sum_{i \geq 2} i(i-1) a_{ij} x^{i-2} y^j) \\ &- 2(\sum_{i\geq 1} i a_{ij} x^{i-1} y^j)(\sum_{j\geq 1} j a_{ij} x^i y^{j-1})(\sum_{j\geq 1} ij a_{ij} x^{i-1} y^{j-1}) \\ &+ (\sum_{i\geq 1} i a_{ij} x^{i-1} y^j)^2 (\sum_{j\geq 2} j(j-1) a_{ij} x^i y^{j-2}) = \sum_{k,m} b_{km} x^k y^m. \end{aligned}$$

Since $F(x,0) = 0$, we have:

$$b_{k0} = 0.$$

Also, using that $a_{k0} = 0$ for $k \geq 0$, an easy calculation gives:

$$b_{k1} = \sum_{m+n+p=k} (p+2)(p+1) a_{m1} a_{n1} a_{p+2,1} - 2 \sum_{m+n+p=k} (m+1)(p+1) a_{m+1,1} a_{n1} a_{p+1,1}.$$

We suppose that the function $F(x,y)$ vanishes at order bigger than 1 along the singular totally geodesic fiber D. In the neighborhood of the point $(0,0)$ the fiber D is given as the intersection of branches $y = 0$ and $x - cy = 0$, hence our assumption implies that

$$F(x,y) = y^2 H(x,y),$$

for some analytic function $H(x,y)$. This implies that all

$$b_{k1} = 0.$$

In particular, from $b_{11} = 0$ using that $a_{01} = 0$, we get:

$$a_{11} = 0.$$

Therefore, we have a contradiction with (4.2) and our assumption that $F(x,y)$ vanishes at order bigger than 1 along a singular geodesic fiber is false, hence the second fundamental form on totally geodesic fibers vanishes of order 1. □

We should underline the fact that the second fundamental form induced by the canonical projective connection on the ball $\mathbb{B}^2$ is invariant under the action of the group $\mathrm{PU}(2,1) \subset \mathrm{PGL}(3)$, as we explained in Section 4.3. Hence, it descends to the ball quotient $X = \mathbb{B}^2/\Gamma$. Moreover, by Lemma 4.9 on the geodesic fibers of the semistable family $f : X \to Y$, the second fundamental form will vanish at order exactely 1.

Theorem 4.10. *If all singular fibers in the family $f : X \to Y$ are totally geodesic, then there is an invertible subsheaf of $f_*\omega_{X/Y}^{\otimes 2}$ which satisfies the maximal case in the Arakelov inequality.*

Proof. The second fundamental form $\beta \in H^0(X, \omega_{X/Y}^{\otimes 2} \otimes f^*T_Y)$, of the fibration $f : X \to Y$ induces a section:

$$\beta_Y \in H^0(Y, f_*\omega_{X/Y}^{\otimes 2} \otimes T_Y).$$

By Lemma 4.6 , β can vanish only on singular fibers, i.e. fibers over the set S. This yields that β_Y can vanish only at points of the set S.
The section $\beta_Y \in H^0(Y, f_*\omega_{X/Y}^{\otimes 2} \otimes T_Y)$ provides an invertible subsheaf

$$\mathcal{F} \subset f_*\omega_{X/Y}^{\otimes 2} \otimes T_Y,$$

and one has:

$$\deg \mathcal{F} = \sum \text{multiplicities of zeros of } \beta_Y.$$

By Lemma 4.9, β vanishes at order 1 along the totally geodesic fibers, so the multiplicities of zeros of the section β_Y will be 1. Hence, we get:

$$\deg \mathcal{F} = \#\mathrm{Zeros}(\beta_Y) \leq \#S.$$

If we suppose that all singular fibers are totally geodesics then β_Y vanishes at every point of S and:

$$\deg \mathcal{F} = \#S.$$

Then, the invertible subsheaf $\mathcal{H} = \mathcal{F} \otimes \omega_Y \subseteq f_*\omega_{X/Y}^{\otimes 2}$, satisfies

$$\deg \mathcal{H} = \deg \mathcal{F} + \deg \omega_Y = \deg \omega_Y(S).$$

The invertible subsheaf $\mathcal{H} \subseteq f_*\omega_{X/Y}^{\otimes 2}$ reaches the bound in the Arakelov inequality. □

The semistable families of curves $f : X \to Y$, where X is a quotient of the complex 2-ball by a torsion-free discrete cocompact subgroup of $\mathrm{PU}(2,1)$, all whose singular fibers are totally geodesic are examples of families whose bicanonical relative sheaf $f_*\omega_{X/Y}^{\otimes 2}$ contains an invertible subsheaf which satisfies the maximal case in the Arakelov inequality.

4.5 Example

In this section we will give examples of semistable families whose geometric variation contains a subvariation whose Higgs field is an isomorphism. In other words, by Theorem 2.22, the base curves in these families are Teichmüller curves. These examples are families $g : W_{\Gamma(N)} \to Y_{\Gamma(N)}$, where the map g is the composition of a ν-cyclic covering over the elliptic modular surface $X_{\Gamma(N)}$ of level N and the elliptic fibration $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$, where $Y_{\Gamma(N)}$ is a modular curve attached to the principal congruence subgroup of level N, for $N \geq 3$. The degree ν of the cyclic covering $W_{\Gamma(N)}$ will depend on N.

Moreover, Livné shows in his thesis [48] that for $\nu = \frac{N}{N-6}$, i.e. for $N \in \{7, 8, 9, 12\}$, the surfaces $W_{\Gamma(N)}$ are of general type with $c_1^2 = 3c_2$. By Yau's result, these surfaces are uniformized by the complex unit 2-ball.

In the previous section we proved that in the case when all singular fibers of a family, uniformized by the complex unit 2-ball, are totally geodesic there is an invertible subsheaf of the direct image of the relative bicanonical sheaf of the family which satisfies the maximal case in the Arakelov inequality. Here, we prove that the pluricanonical relative sheaf of the families $g : W_{\Gamma(N)} \to Y_{\Gamma(N)}$ contains an invertible subsheaf which satisfies the maximal case in the Arakelov inequality. Later, we will prove that in the case when $N \in \{7, 8, 9, 12\}$ the singular fibers are totally geodesic, as expected.

4.5.1 Preliminaries

Let Γ be a torsion free subgroup of finite index of the group $\mathrm{SL}_2(\mathbb{Z})$. The group Γ acts discretely on $\mathbb{H}$. The quotient $\mathbb{H}/\Gamma$ is a non-compact Riemann surface. Adding a finite number of cusps to $\mathbb{H}/\Gamma$ we get a compact Riemann surface Y_Γ, called the modular curve attached to Γ.

Definition 4.19. ([64] §1.4 or [65] §4) A point $y \in Y_\Gamma$ is called a cusp of width b if one of its representing points $z \in \mathbb{Q} \cup \{\infty\}$ has the stabilizer generated by an element which is conjugate in $\mathrm{SL}_2(\mathbb{Z})$ to either $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$, for $b > 0$. Respectively, y is called a cusp of the first or of the second kind.

We say that a point $y \in Y_\Gamma$ is an elliptic point if y is not a cusp and its representing point $z \in \mathbb{H}$ has the stabilizer Γ_z in Γ, generated by an element of order 3, which is conjugate in $\mathrm{SL}_2(\mathbb{Z})$ to either $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

We will use the notations:

- μ is the index of the projectivization of Γ in $\mathrm{PSL}(2, \mathbb{Z})$;
- t_1 is the number of cusps of the first kind in Y_Γ;
- t_2 is the number of cusps of the second kind in Y_Γ;
- $t' = t_1 + t_2$ is the number of cusps in Y_Γ;
- s is the number of elliptic points in Y_Γ;
- $t = t' + s$.

Lemma 4.11. *(§4[65]) The genus g of Y_Γ satisfies:*

$$2g - 2 + t' + \frac{2}{3}s = \frac{1}{6}\mu.$$

Definition 4.20. Let $f : X \to Y$ be a projective family of curves, smooth over $U = Y \backslash S$. The surface X is called an elliptic fibration or an elliptic surface if:

1. all smooth fibers $f^{-1}(y), y \in U$ are elliptic curves, and singular fibers contain no (-1)-curves;
2. there exists a holomorphic map $\varphi_0 : Y \to X$ such that $f \circ \varphi_0 = id_Y$, called a zero section of f.

Following Chapter 12 from [44] and Chapter 2 from [65], we give several facts about elliptic fibrations over a modular curve.

($\mathbf{A}_1$) The quotient of $\mathbb{H} \times \mathbb{C}$ by automorphisms of the form:

$$(\tau, z) \to (\gamma(\tau), \frac{z + m\tau + n}{c\tau + d}),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, (m, n) \in \mathbb{Z}^2$, defines a surface equipped with a morphism to the modular curve Y_Γ. The fiber over the image in Y_Γ of a general point $\tau \in \mathbb{H}$ is the elliptic curve corresponding to the lattice $\mathbb{Z} \oplus \mathbb{Z}\tau$. The surface obtained in this way can be extended to an elliptic surface X_Γ over the modular curve Y_Γ. In this fibration singular fibers lie over cusps and elliptic points of the modular curve Y_Γ. The surface X_Γ is called the elliptic modular surface attached to the group Γ.

($\mathbf{A}_2$) Let us denote by $\varphi_\Gamma : Y_\Gamma \to X_\Gamma$ a section of the elliptic modular fibration $f : X_\Gamma \to Y_\Gamma$. By Theorem 6.8 from [62] one has:

$$\omega_{X_\Gamma} = f^*(\omega_{Y_\Gamma} \otimes \mathcal{L}^{-1}), \tag{4.3}$$

for some line bundle $\mathcal{L}$ such that $\deg \mathcal{L}^{-1} = \chi(\mathcal{O}_{X_\Gamma})$. Moreover, one gets that K_{X_Γ} is a vertical divisor, i.e. it is linearly equivalent to a smooth fiber of the family and one has:

$$K_{X_\Gamma} \approx (2g - 2 + \chi(\mathcal{O}_{X_\Gamma}))C,$$

where C is a smooth fiber. Hence, one gets:

$$K_{X_\Gamma} \cdot K_{X_\Gamma} = 0,$$

or

$$c_1^2(X_\Gamma) = 0.$$

On the other side, the Noether formula:

$$12\chi(\mathcal{O}_{X_\Gamma}) = c_1^2(X_\Gamma) + c_2(X_\Gamma)$$

yields

$$12(1 - q + p_g) = 12 \deg \mathcal{L}^{-1} = \deg c_2(X_\Gamma) = \chi(X_\Gamma), \tag{4.4}$$

where q is the irregularity of X_Γ and p_g is geometric genus of X_Γ.

Using Lemma VI.4 from [4] which describes the Euler characteristic of a fibration of a surface one gets:

$$\chi(X_\Gamma) = \chi(Y_\Gamma)\chi(C) + \sum_{t\in T}(\chi(X_{\Gamma,t}) - \chi(C)),$$

where C is a smooth fiber in the family, the set T is the discriminant locus of the family and $X_{\Gamma,t}$ is the singular fiber corresponding to the point $t \in T$. The Euler characteristic of an elliptic curve vanishes and one gets:

$$c_2(X_\Gamma) = \sum_{t\in T}\chi(X_{\Gamma,t}). \tag{4.5}$$

By Theorem 12.2 from [44] and Proposition 4.2 from [65] we get:

$$c_2(X_\Gamma) = \sum_{t\in T}\chi(X_{\Gamma,t}) = \mu + 6t_2 + 8s. \tag{4.6}$$

Therefore, the Euler caracteristic of the modular elliptic surface X_Γ is positive and by (4.4) one gets that $\chi(\mathcal{O}_{X_\Gamma})$ is positive. In particular, the degree of the line bundle $\mathcal{L}^{-1}$ is positive, hence it is an ample divisor on Y_Γ.

One should note that:

$$p_g = h^0(X, \omega_{X_\Gamma}) = h^0(X, f^*(\omega_{Y_\Gamma}\otimes\mathcal{L}^{-1})) = h^0(Y, \omega_{Y_\Gamma}\otimes\mathcal{L}^{-1}).$$

Using the Rieman-Roch formula, the Serre duality and the fact that $h^0(Y,\mathcal{L}) = 0$ (since $\mathcal{L}^{-1}$ is ample on Y) one gets:

$$h^0(Y, \omega_{Y_\Gamma}\otimes\mathcal{L}^{-1}) = \deg(\omega_{Y_\Gamma}\otimes\mathcal{L}^{-1}) - g + 1, \tag{4.7}$$

i.e.

$$p_g = \deg(\omega_{Y_\Gamma}\otimes\mathcal{L}^{-1}) - g + 1 = g - 1 - \deg\mathcal{L}, \tag{4.8}$$

where g is the genus of Y_Γ.

By (4.4) and (4.8) we get:

$$q = p_g + 1 - \frac{c_2(X_\Gamma)}{12} = g - 1 - \deg\mathcal{L} + 1 + \deg\mathcal{L} = g.$$

Hence, the irregularity of the elliptic modular surface is equal to the genus of the modular curve Y_Γ.

($\mathbf{A}_3$) By Lemma (4.11) one has:

$$\mu = 12g - 12 + 6t_1 + 6t_2 + 4s.$$

As we saw before by (4.6) one has:

$$c_2(X_\Gamma) = 12 \deg \mathcal{L}^{-1} = \mu + 6t_2 + 8s,$$

hence

$$\deg \mathcal{L}^{-1} = g - 1 + \frac{1}{2}t_1 + t_2 + s.$$

Then by (4.8) we get:

$$p_g = 2g - 2 + \frac{1}{2}t_1 + t_2 + s. \tag{4.9}$$

Definition 4.21. The principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level N is defined to be:

$$\Gamma(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | a, d \equiv 1 (\mathrm{mod} N); b, c \equiv 0 (\mathrm{mod} N) \right\}.$$

The curve $Y_{\Gamma(N)}$ is called the modular curve of level N. The elliptic modular surface $X_{\Gamma(N)}$ attached to the group $\Gamma(N)$ is called the elliptic modular surface of level N.

From now on, we will suppose that $N \geq 3$. Following results from [37] and the previous results about elliptic modular surfaces, we will list here several properties of the family $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$.

($\mathbf{B}_1$) The group $\Gamma(N)$ is torsion free, so $s(N) = 0$.

($\mathbf{B}_2$) All cusps are of the first kind, hence $t_2(N) = 0$ and $t(N) = t_1(N)$. The set of cusps will be denoted by T. This set is the discriminant locus of the family $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$.

($\mathbf{B}_3$) $t(N) = \frac{\mu(N)}{N}$, where $\mu(N) = \dfrac{1}{2}N^3 \prod_{p|N}(1 - \dfrac{1}{p^2})$, where the product is taken over prime numbers p;

($\mathbf{B}_4$) From point ($\mathbf{A}_3$), the genus of the curve $Y_{\Gamma(N)}$ is given by:

$$g(N) = 1 + \frac{(N-6)\mu(N)}{12N}.$$

It is equal to the irregularity $q(N)$ (see ($\mathbf{A}_2$)). The geometric genus of the surface $X_{\Gamma(N)}$ is given by:

$$p_g(N) = \frac{N-3}{6N}\mu(N).$$

($\mathbf{B}_5$) The elliptic modular surface $X_{\Gamma(N)}$ of level N has N^2 disjoint sections over the base curve $Y_{\Gamma(N)}$. These sections will be denoted by D_i and the sum of these sections is $D = \sum_{i=1}^{N^2} D_i$. For $N \geq 4$, the divisor $D = \sum_{i=1}^{N^2} D_i$ is divisible in $\mathrm{Pic}(X_{\Gamma(N)})$ by N if N is odd or by $\dfrac{N}{2}$ otherwise.

($\mathbf{B}_6$) From point ($\mathbf{A}_2$), the canonical bundle of $X_{\Gamma(N)}$ is given as

$$\omega_{X_{\Gamma(N)}} = f^*(\omega_{Y_{\Gamma(N)}} \otimes \mathcal{L}^{-1}),$$

where $\mathcal{L}$ is a line bundle on $Y_{\Gamma(N)}$ such that

$$\deg \mathcal{L}^{-1} = p_g(N) - q(N) + 1 = \frac{\mu(N)}{12}.$$

($\mathbf{B}_7$) Non-singular fibers of the family $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$ are non-singular elliptic curves. The divisor D intersects a smooth fiber in N^2 points. Singular fibers of the family are above the cusps of $Y_{\Gamma(N)}$. A singular fiber F of the family is of the form $F = \sum_{i=0}^{N-1} F_i$, where F_i are non-singular rational curves with $F_i^2 = -2$. A curve F_i intersects transversely curves F_{i-1} and F_{i+1}. Hence, singularities of the divisor F are nodes. The $N-$cycle of these curves is such that every F_i intersects the divisor D in N points, i.e. every curve F_i intersects exactly N sections. The points of intersections of divisors F and D are not the nodes of divisor F. It is obvious that the family $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$ is semistable. Then by Lemma 1.9 one gets:

$$\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}} = \Omega^1_{X_{\Gamma(N)}/Y_{\Gamma(N)}}(\log f^*(T)).$$

4.5.2 The construction

We will suppose that $N \geq 4$. Let $\nu \geq 2$ be an integer such that ν divides N if N is odd or ν divides $\dfrac{N}{2}$ if N is even. Hence, by ($\mathbf{B}_5$), there exists a line bundle $\mathcal{M}$ in $\mathrm{Pic}(X_{\Gamma(N)})$ such that:

$$\mathcal{O}_{X_{\Gamma(N)}}(D) = \mathcal{M}^\nu.$$

Now, we can construct a cyclic covering $W_{\Gamma(N)}$ over $X_{\Gamma(N)}$ of degree ν, ramified along the divisor D. Since all components of D are disjoint, D is a smooth divisor. Therefore, the cyclic covering $\tau : W_{\Gamma(N)} \to Y_{\Gamma(N)}$ ramified along D is smooth, see Lemma 1.6. The induced family $g = f \circ \tau : W_{\Gamma(N)} \to Y_{\Gamma(N)}$, has singular fibers over the set of cusps T of the curve $Y_{\Gamma(N)}$, which is also the discriminant locus for the family $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$. There is no fibers of the family $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$ in the branched locus D of the covering, so all fibers of the family $g : W_{\Gamma(N)} \to Y_{\Gamma(N)}$ are reduced, on the other side the branch divisor D does not contain singular points (nodes) of singular fibers of the family $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$ (see $(\mathbf{B}_7)$), hence $\tau^*(D) + g^*(T)$ is a normal crossing divisor, see Remark 1.14. We have that all fibers of $g : W_{\Gamma(N)} \to Y_{\Gamma(N)}$ are reduced, singular fibers are normal crossing divisors and they do not contain exceptional curves of first kind, hence the family $g : W_{\Gamma(N)} \to Y_{\Gamma(N)}$ is semistable.

The semistability of the family $g : W_{\Gamma(N)} \to Y_{\Gamma(N)}$ and Lemma 1.9 yield that:

$$\omega_{W_{\Gamma(N)}/Y_{\Gamma(N)}} = \Omega^1_{W_{\Gamma(N)}/Y_{\Gamma(N)}}(\log(g^*(T))).$$

By Lemma 1.16 one has:

$$\tau_*\omega_{W_{\Gamma(N)}/Y_{\Gamma(N)}} = \omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}} \oplus \bigoplus_{i=1}^{\nu-1} \Omega^1_{X_{\Gamma(N)}/Y_{\Gamma(N)}}(\log(f^{-1}(T) + \Gamma_i)) \otimes \mathcal{M}^{(-i)},$$

where Γ_i is the sum of components of D whose multiplicities multiplied by i are not divisible by ν. Then it is plain to see that $\Gamma_i = D$, for all $i = 1, ..., \nu-1$, since D is a reduced divisor. We get: $\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}} \subset \tau_*\omega_{W_{\Gamma(N)}/Y_{\Gamma(N)}}$, or

$$f_*\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}} \subset g_*\omega_{W_{\Gamma(N)}/Y_{\Gamma(N)}}. \tag{4.10}$$

Lemma 4.12. *The sheaf $f_*\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}}$ is an invertible sheaf and for any positive integer n the sheaf*

$$\mathcal{H} = (f_*\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}})^{\otimes n}$$

is a subsheaf of $g_*\omega^{\otimes n}_{W_{\Gamma(N)}/Y_{\Gamma(N)}}$ and satisfies the maximal case in the Arakelov inequality for the semistable family $g : W_{\Gamma(N)} \to Y_{\Gamma(N)}$, i.e.

$$\deg \mathcal{H} = \frac{n}{2} \deg \omega_{Y_{\Gamma(N)}}(T).$$

Proof. As $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$ is an elliptic fibration, the canonical bundle of a generic fiber C is trivial, i.e. $\omega_C = \mathcal{O}_C$ which yields $h^0(C, \omega_C) = 1$. This means that the rank of the sheaf $f_*\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}}$ is 1, so it is an invertible sheaf. Also, by $(\mathbf{B}_6)$ one has:

$$\begin{aligned}\deg f_*\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}} &= \deg f_*(\omega_{X_{\Gamma(N)}} \otimes f^*\omega^{-1}_{Y_{\Gamma(N)}}) = \deg(f_*\omega_{X_{\Gamma(N)}} \otimes \omega^{-1}_{Y_{\Gamma(N)}}) \\ &= \deg(\omega_{Y_{\Gamma(N)}} \otimes \mathcal{L}^{-1} \otimes \omega^{-1}_{Y_{\Gamma(N)}}) = \deg \mathcal{L}^{-1} = \frac{\mu(N)}{12}.\end{aligned}$$

On the other hand, one has:

$$\deg \omega_{Y_{\Gamma(N)}}(T) = 2g(N) - 2 + t = \frac{\mu(N)}{6}.$$

By (4.10) we have:

$$\mathcal{H} = (f_*\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}})^{\otimes n} \subset (g_*\omega_{W_{\Gamma(N)}/Y_{\Gamma(N)}})^{\otimes n} \subseteq g_*\omega^{\otimes n}_{W_{\Gamma(N)}/Y_{\Gamma(N)}}$$

and

$$\deg \mathcal{H} = n f_*\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}} = \frac{n\mu(N)}{12} = \frac{n}{2}\deg \omega_{Y_{\Gamma(N)}}(T).$$

So, for the semistable family $g : W_{\Gamma(N)} \to Y_{\Gamma(N)}$, with discriminant locus the set of cusps T, the invertible sheaf $\mathcal{H} = (f_*\omega_{X_{\Gamma(N)}/Y_{\Gamma(N)}})^{\otimes n} \subset g_*\omega^{\otimes n}_{W_{\Gamma(N)}/Y_{\Gamma(N)}}$ satisfies the case of the equality in the Arakelov inequality. □

Proposition 4.13. *For the family $g : W_{\Gamma(N)} \to Y_{\Gamma(N)}$, the curve $Y_{\Gamma(N)} \backslash T$ is a Teichmüller curve.*

Proof. This is a consequence of Lemma 3.13 for $n = 1$.

□

Livné showed in his thesis [48] §1.6 that the surfaces $W_{\Gamma(N)}$ with $\nu = \frac{N}{N-6}$, i.e. $N = 7, 8, 9, 12$, satisfy $c_1^2 = 3c_2$. Due to well known Yau's result [84], the surfaces with $c_1^2 = 3c_2$ are quotients of the complex unit 2-ball by a discrete, co-compact, torsion-free subgroup of $\mathrm{PU}(2, 1)$.

In order to prove that all singular fibers in the semistable families:

$$g : W_{\Gamma(N)} \to Y_{\Gamma(N)} \text{ for } N \in \{7, 8, 9, 12\},$$

are totally geodesic let us state and prove one auxiliary lemma which gives an approach for detecting the totally geodesic curves on a smooth complex 2-ball quotient.

Lemma 4.14. *Suppose D is a reduced (not necessarily irreducible) curve on a smooth complex two-ball quotient X self-intersecting only at k distinct points with simple multiplicities given by $(b_1, ..., b_k)$ and let us denote by D_i ($i = 1, 2..., n$) its irreducible components, $\hat{D}_i$ their normalization. Let $\alpha : \hat{D} = \cup_i \hat{D}_i \to D$ be the normalization of D. Then D is totally geodesic if and only if*

$$K_X \cdot D = 3\sum_{i=1}^{n}(g(\hat{D}_i) - 1).$$

Proof. The direction when D is totally geodesic is Lemma 6 in [11]. Here, we will prove the other direction. We suppose that:

$$K_X \cdot D = 3\sum_{i=1}^{n}(g(\hat{D}_i) - 1).$$

Note that we also have:

$$K_X \cdot D = \int_D c_1(K_X) = \int_{\hat{D}} \alpha^* c_1(K_X) = \int_{\hat{D}} c_1(\alpha^* K_X),$$

and

$$3\sum_{i=1}^{n}(g(\hat{D}_i)-1)=\frac{3}{2}\sum_{i=1}^{n}\deg K_{\hat{D}_i}=\frac{3}{2}\sum_{i=1}^{n}\int_{\hat{D}_i}c_1(K_{\hat{D}_i}).$$

This yields:

$$\int_{\hat{D}}c_1(\alpha^*K_X)=\frac{3}{2}\int_{\hat{D}}c_1(K_{\hat{D}}). \tag{4.11}$$

By [9] §1 or Lemma 4.6 we have:

$$c_1(\alpha^*K_X)=3\alpha^*\xi,$$

$$c_1(K_{\hat{D}})=2\alpha^*\xi+\alpha^*\sigma,$$

where ξ is the Kähler form of the metric on X and σ is (1,1)-form, non negative definite at every point of D. It is explained in §1 of [9] that if σ vanishes identically on D, then the second fundamental form vanishes on D, or equivalently D is totally geodesic. Using that σ is non-negative and equality 4.11, one gets:

$$\int_{\hat{D}}3\alpha^*\xi=\frac{3}{2}\int_{\hat{D}}(2\alpha^*\xi+\alpha^*\sigma),$$

hence $\alpha^*\sigma=0$ on $\hat{D}$. As a consequence one has that the irreducible components of D are totally geodesic curves. □

Lemma 4.15. *In the notations from the beginning of the section, for* $\nu=\frac{N}{N-6}$, *the singular fibers in the family*

$$g: W_{\Gamma(N)}\to Y_{\Gamma(N)}$$

are totally geodesic.

Proof. The singular fibers of the fibration $g: W_{\Gamma(N)}\to Y_{\Gamma(N)}$ are the fibers over cusps of $Y_{\Gamma(N)}$ and they are $\nu-$cyclic coverings of singular fibers of the family $f: X_{\Gamma(N)}\to Y_{\Gamma(N)}$. Recalll that the map $\tau: W_{\Gamma(N)}\to X_{\Gamma(N)}$ is the ν-cyclic covering ramified along the divisor $D=\sum_{i=1}^{N^2}D_i$.

For a singular fiber of the family $f: X_{\Gamma(N)}\to Y_{\Gamma(N)}$, one has:

$$F=\sum_{i=0}^{N-1}F_i,$$

where $F_i\cong\mathbb{P}^1_{\mathbb{C}}$. Then $\tau^*(F)=C=\sum_{i=0}^{N-1}C_i$, where the C_i's are $\nu-$cyclic coverings of $F_i\cong\mathbb{P}^1$. The cyclic coverings C_i's of $\mathbb{P}^1$ are ramified over N points of F_i, which are not nodes of F. This holds by $(\mathbf{B}_7)$. Hence, by the Riemann-Hurwitz formula one gets:

$$\deg K_{C_i}=\nu\deg K_{\mathbb{P}^1_{\mathbb{C}}}+N(\nu-1)=-2\nu+N(\nu-1),$$

and
$$g(C_i) = \frac{(\nu - 1)(N - 2)}{2}.$$
In Livné's thesis §1.5 we can find that the canonical divisor of the surface $W_{\Gamma(N)}$ is given by:

$$K_{W_{\Gamma(N)}} = (\nu - 1)\sum_{i=1}^{N^2} \tilde{D}_i + \frac{N-4}{4N}\mu(N)\tilde{\Phi},$$

where $\tilde{\Phi} = \tau^*(\Phi)$, for Φ a smooth fiber in the family $f : X_{\Gamma(N)} \to Y_{\Gamma(N)}$ and $\tilde{D}_i = \tau^* D_i$.

Using the fact that fibers in a family do not intersect, we have $F \cdot \Phi = 0$, then $\tau^* F \cdot \tau^* \Phi = 0$, i.e.
$$C \cdot \tilde{\Phi} = 0.$$
Again by $(\mathbf{B}_7)$ and the fact that C_i is ramified at N points one gets

$$C_i \cdot \sum_{j=1}^{N^2} \tilde{D}_j = N.$$

Bringing together all these facts we have:

$$K_{W_{\Gamma(N)}} \cdot C_i = \left((\nu - 1)\sum_{i=1}^{N^2} \tilde{D}_i + \frac{N-4}{4N}\mu(N)\tilde{\Phi} \right) \cdot C_i = (\nu - 1)N,$$

and
$$3\sum_{i=1}^{N} (g(C_i) - 1) = 3N\frac{N\nu - 2\nu - N}{2}.$$
Now, for $\nu = \frac{N}{N-6}$ we get:

$$K_{W_{\Gamma(N)}} \cdot C = N(K_{W_{\Gamma(N)}} \cdot C_i) = 6\frac{N^2}{N-6},$$

and
$$3\sum_{i=1}^{N} (g(C_i) - 1) = 6\frac{N^2}{N-6},$$
which yields that C is totally geodesic by the previous lemma.

□

The elliptic modular surface $X_{\Gamma(12)}$ has 144 sections which form the divisor D. This divisor is divisible in the group $\mathrm{Pic}(X_{\Gamma(12)})$ by 6. Hence, we can construct the cyclic covering $W_{\Gamma(12)}$ over $X_{\Gamma(12)}$ ramified over the divisor D of degree $\nu = \dfrac{12}{12-6} = 2$. Therefore, $W_{\Gamma(12)}$ is a quotient of a complex 2-ball by a torsion free cocompact discrete subgroup of $\mathrm{PU}(2,1)$. All singular fibers of the family $g : W_{\Gamma(12)} \to Y_{\Gamma(12)}$, i.e. the fibers over the set of cusps T of $Y_{\Gamma(12)}$ are totally geodesic. Let us calculate the genus of smooth fibers $W_y(N)$ in the family. By Hurwitz-Riemann one gets:

$$g(W_y(N)) = \frac{144+2}{2} = 73,$$

since the divisor D intersect a smooth fiber of the family $f : X_{\Gamma(12)} \to Y_{\Gamma(12)}$ (an elliptic curve) in $N^2 = 144$ points. The invertible sheaf $f_*\omega_{X_{\Gamma(12)}/Y_{\Gamma(12)}}$ is a subsheaf of $g_*\omega_{W_{\Gamma(12)}/Y_{\Gamma(12)}}$ and it satisfies the maximal case in the Arakelov inequality. The curve $Y_{\Gamma(12)}\backslash T$ is a Teichmüller curve in $\mathcal{M}_{73}$, the moduli space of curves of genus 73, for the family $g : W_{\Gamma(12)} \to Y_{\Gamma(12)}$.

In the case when $N \in \{7,8,9\}$ we get $\nu \in \{7,4,3\}$. The invertible sheaves which satisfy the maximal case in the Arakelov inequalities are $f_*\omega_{X_{\Gamma(7)}/Y_{\Gamma(7)}}$, $f_*\omega_{X_{\Gamma(8)}/Y_{\Gamma(8)}}$ and $f_*\omega_{X_{\Gamma(9)}/Y_{\Gamma(9)}}$. The curves $Y_{\Gamma(7)}\backslash T$, $Y_{\Gamma(8)}\backslash T$ and $Y_{\Gamma(9)}\backslash T$ are Teichmüller curves in $\mathcal{M}_{148}$, $\mathcal{M}_{97}$ and $\mathcal{M}_{82}$, for the families $g : W_{\Gamma(7)} \to Y_{\Gamma(7)}$, $g : W_{\Gamma(8)} \to Y_{\Gamma(8)}$ and $g : W_{\Gamma(9)} \to Y_{\Gamma(9)}$, respectively.

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Samenvatting

Dit proefschrift is onderverdeeld in vier hoofdstukken. De eerste twee hoofdstukken zijn van een inleidende aard, met een aantal bekende resultaten met licht aangepaste bewijzen. Het derde hoofdstuk biedt een nieuw gezichtspunt op het bewijs van ongelijkheid (8) uit de Inleiding en maakt een aantal opmerkingen over het geval van gelijkheid in (8). Het laatste hoofdstuk bevat originele resultaten, en geeft een aantal voorbeelden van families waarin gelijkheid in (8) wordt behaald.

Het eerste hoofdstuk behandelt elementaire definities en resultaten aangaande cyclische overdekkingen van n-differentieerbare variëteiten, logaritmische differentiaalvormen, en de cohomologie van cyclische overdekkingen. Hierbij volgen we het boek [20] van Hélène Esnault en Eckart Viehweg op de voet. Sommige resultaten worden in hun oorspronkelijke vorm besproken, andere worden opnieuw bewezen en aangepast aan de context van dit proefschrift.

In het tweede hoofdstuk brengen we de definities en constructies in herinnering van Higgs-bundels en logaritmische Higgs-bundels afkomend van variaties van Hodge-structuren op compacte en niet-compacte krommen, alsmede een aantal belangrijke resultaten van Deligne en Simpson. In de laatste sectie van dit hoofdstuk geven we een kort résumé van elementaire feiten aangaande de Teichmüller-ruimte en Teichmüller-krommen. Ook bespreken we een resultaat van Möller over de samenhang tussen gepolariseerde variaties van rang 2 en gewicht 1, en Teichmüller-krommen.

Het derde hoofdstuk is het belangrijkste hoofdstuk, in de technische zin. Hier bespreken we het bewijs dat Viehweg en Zuo hebben gegeven van de Arakelov-ongelijkheid (8). Hun oorspronkelijke resultaat geldt voor semistabiele families van n-variëteiten, maar in dit proefschrift passen we het bewijs aan, en vullen we alle details in, voor het geval van families van semistabiele krommen. In de laatste sectie van dit hoofdstuk bespreken we het geval dat gelijkheid optreedt in (8). We noemen dit het “maximale geval”. We hadden aanvankelijk gehoopt dat we zouden kunnen bewijzen dat voor $\nu \geq 2$ de kromme $Y \setminus S$ een Teichmüller-kromme is. Helaas hebben we dit niet kunnen bewijzen, en kunnen we slechts gedeeltelijke informatie leveren over dergelijke families.

Het laatste hoofdstuk van dit proefschrift bevat originele resultaten. Het lijkt erop dat voorbeelden van semistabiele families van krommen waarvoor het directe beeld van de relatieve plurikanonieke schoof een inverteerbare schoof bevat waarvoor gelijkheid in (8) optreedt, niet breed bekend zijn, behalve in de voor de hand liggende gevallen die kunnen worden geconstrueerd voor $\nu = 1$.

Het blijkt dat voor zekere families van krommen die door de complexe eenheidsbal worden geüniformiseerd, er een natuurlijke inverteerbare deelschoof bestaat van het directe beeld van de relatieve bikanonieke schoof. Deze deelschoof kan worden geconstrueerd met behulp van de tweede fundamentaalvorm van de familie. We geven een beschrijving van deze deelschoof met behulp van Mok's resultaat over projectieve tweede fundamentaalvormen en tautologische foliaties op de projectivizering van de raakbundel van vormen van de complex hyperbolische ruimte. Dit zijn quotiënten van de complexe n-eenheidsbal $\mathbb{B}^n$ langs een discrete, co-compacte, torsievrije ondergroep van $\mathrm{PU}(n, 1)$.

Stelling *Zij $f\colon X = \mathbb{B}^2/\Gamma \to Y$ een semistabiele familie van krommen, waarbij Γ een discrete, co-compacte, torsievrije ondergroep van $PU(2, 1)$ is. We nemen aan dat de familie glad is over $Y \setminus S$, dat alle singuliere vezels totaal geodetisch zijn en dat het geslacht van Y groter is dan 1. Dan bestaat er een inverteerbare deelschoof van het directe beeld van de relatieve bikanonieke schoof $f_*\omega_{X/Y}^{\otimes 2}$ waarvoor de Arakelov-gelijkheid geldt in (8).*

Naar aanleiding van dit resultaat en de resultaten van Livné uit [49] presenteren we een aantal voorbeelden van semistabiele families van krommen geüniformiseerd door de complexe 2-bal over modulaire krommen van niveau $N \in \{7, 8, 9, 12\}$. We bewijzen dat alle singuliere vezels in deze families totaal geodetisch zijn. Daarna bewijzen we dat dit voorbeelden zijn van families waarvan de basiskromme, zonder de discriminantlocus, een Teichmüllerkromme is. We merken echter op dat deze voorbeelden van families waarvoor het directe beeld van de relatieve bikanonieke schoof een maximale inverteerbare deelschoof bevat opnieuw afkomen van maximale gevallen waarbij $\nu = 1$.

Een vraag die nog beantwoord moet worden is of elke familie zoals in bovenstaande Stelling een voorbeeld geeft van een Teichmüller-kromme.

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Curriculum Vitae

Nikola Damjanovic was born on February 15, 1988 in Cetinje, Montenegro. He attended elementary school M.M.Burzan and mathematical high school S.Skerovic in Podgorica. After that, he entered the University of Montenegro in 2007 where he got his bachelor degree in mathematics and computer science in 2010.

After finishing his bachelor studies, he joined the ALGANT Master program. He spent the first year of the program at the University of Bordeaux, and went on to spend the second year at the Univeristy of Milan. He graduated from the program in 2012, with his master thesis "Automorphisms on K3 surfaces", which was supervised by Prof. Lambertus Van Geemen. He obtained a joint master diploma from the University of Bordeaux and the University of Milan.

Following master studies, Nikola worked at the rectorate of the University of Montenegro and then he started ALGANT PhD program, working on a joint project between the University of Bordeaux and the University of Leiden, supervised by professors Vincent Koziarz and Robin de Jong. During his PhD thesis Nikola has been working on a subject connected to the theory of Arakelov inequalities, Higgs bundles and ball quotients. As a PhD student Nikola has been teaching at the University of Bordeaux. Nikola is now living in Bordeaux, where he works for a french IT company.